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Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. MME-7
Theory of Electromagnetic Research
Contract No. DA49-170-sc-2253
November, 1957

Diffraction in Inhomogeneous Media

BERNARD D. SECKLER and JOSEPH B. KELLER

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Abstract

Part I - The Geometric Theory of Diffraction

In Part I, a geometric method is given for finding the field in inhomogeneous media containing smooth convex bodies. The field due to a plane wave in an unbounded medium is constructed by introducing complex rays in the refraction shadow. By extending Fermat's principle, we explain the occurrence of certain diffracted rays in the boundary problems. By then modifying the law of conservation of energy, we obtain the field along the diffracted rays in addition to the field in the geometric lit region. Certain diffraction coefficients and decay exponents are introduced and general formulas are obtained for them. The theory is then applied to several problems in which the medium is plane or cylindrically stratified and the boundary planar or circular.

Part II - Asymptotic Expansion of Solutions of Boundary Value Problems

In Part II, we determine expressions for the exact solution to various boundary value problems corresponding to the special problems of Part I. Asymptotic forms are obtained by using the method of stationary phase in conjunction with the WKB method and Watson transformations. In all cases, we compare the results of Part I with these asymptotic expansions and find that the two are in agreement.

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Part I. The Geometrical Diffraction Theory

1. Introduction

In the analysis of wave propagation in inhomogeneous media the method of geometrical optics is often employed. Not only is it used to determine where waves propagate, but also to evaluate the field quantitatively. However it fails to yield any field other than zero in shadow regions, i.e., regions devoid of rays. It is our purpose to show that the geometrical theory of diffraction[1], a recent extension of geometrical optics, is capable of accurately describing the diffracted field in shadows. It also yields diffraction corrections to the geometrical optics field in illuminated regions.

This theory differs from ordinary geometrical optics by the inclusion of certain additional rays called diffracted rays and others called complex or imaginary rays. The former are associated with the boundaries or discontinuities of the medium while the latter are associated with refraction shadows which occur in the absence of boundaries. In addition to characterizing these new rays, the theory also provides a method for calculating the field associated with them.

In Part I of this article we explain the theory and apply it to a number of problems. Then in Part II we reconsider some of these same problems from the viewpoint of boundary value problems for partial differential equations. We choose cases which can be solved exactly by the method of separation of variables. Then we evaluate these solutions asymptotically for large values of ka ($k = \frac{2\pi}{\lambda}$ where λ is the wavelength, a = typical dimension in the problem). Upon comparing the results of Parts I and II we verify that the geometrical theory is indeed correct, at least in these cases.

The geometrical theory of diffraction has previously been applied to diffraction by convex cylinders[2], diffraction by apertures in thin screens[3], diffraction by smooth opaque objects[4], diffraction in waveguides[14], and other problems[15]. In all cases the results of this theory coincided with the asymptotic expansions for large ka, of exact solutions of boundary value problems, when these were available for comparison. The results were also accurate for wavelengths as large as the dimensions of the scatterer.

2. Ray tracing

In geometrical optics a medium is characterized by its index of refraction $n(x)$. In terms of $n(x)$ a ray is defined as a curve $x(t)$ which satisfies the equation

$$\ddot{x} = \nabla(n^2/2) \qquad . \tag{1}$$

The parameter t is normalized by the condition

$$\dot{x}^2 = n^2 \qquad . \tag{2}$$

It is to be noted that x denotes a vector so that $\dot{x}^2$ denotes the square of the length of $\dot{x}$. When a ray hits an interface or boundary it gives rise to reflected and refracted rays. The initial directions of these rays are determined by the laws of reflection and refraction, and their subsequent paths are determined by (1) and (2).

The rays described above, and the multiply reflected and refracted rays to which they give rise, constitute all the rays of geometrical optics. In the geometrical theory of diffraction we introduce additional rays of various kinds called diffracted rays. The only type of diffracted ray which we will consider here is the kind produced when a ray is tangent to a boundary surface. We assume that such a ray splits at the point of tangency. One part of the ray continues along the original path of the incident ray. The other part proceeds along the boundary as a surface ray. At each point the surface ray splits again, shedding a ray which leaves the surface in the direction of the tangent to the surface ray. This shed ray then proceeds in accordance with (1) and (2). The path of the diffracted ray along the surface is an optical geodesic. By this we mean that the surface ray between two points Q_1 and P_1 on the surface is a curve which renders stationary the optical length $\int_{Q_1}^{P_1} n[x(s)]ds$ among all curves on the surface. Here s denotes arclength.

The above characterization of a diffracted ray is equivalent to the following modified form of Fermat's principle: A diffracted ray connecting two points Q and P is a curve which renders stationary the optical length $\int_Q^P n[x(s)]ds$ among all curves joining Q and P and having an arc on the boundary. From this principle it follows that the diffracted ray consists of two segments of ordinary rays which satisfy (1) and (2), one from Q tangent to the boundary and another from P tangent to the boundary. The two points of tangency are connected by a surface geodesic which is tangent to the two ordinary ray segments. These are just the properties of diffracted rays described above.

In determining a surface geodesic the index of refraction associated with one of the two sides of the surface must be chosen. We will consider, for simplicity, only those surfaces which bound the region within which propagation occurs. Therefore $n(x)$ will be defined on only one side of the surface, so no ambiguity can arise.

In addition to the diffracted rays, we will also consider complex or imaginary rays. These are defined to be complex valued solutions $x(t)$ of the ray equations (1) and (2). Such rays can be defined when $n(x)$ is an analytic function of x, and also in certain cases, when n is piece-wise analytic. The particular complex rays which arise in any problem are determined by the incident rays. This will become clear in Section 4, where these rays are utilized.

3. Field calculation

In the present theory, as in geometrical optics, a field $u(P)$ is associated with each point P on a ray. This field is composed of a phase $\phi(s)$ and an amplitude A(s), both of which may vary with arclength s along the ray. The phase is given in terms of optical length by the formula

$$\phi(s) = \phi_0 + \int_0^s n[x(s)]\,ds \quad . \tag{3}$$

Here ϕ_0 is the phase at the point Q from which arclength s along the ray is measured. The direction of increasing s must be the direction of propagation along the ray.

The amplitude A(s) is determined by the principle of conservation of energy. The optical form of this principle states that in a time periodic (harmonic) field the energy flux is the same through all cross sections of a tube of rays. Upon applying this principle to a narrow tube of rays containing the ray from Q to P we obtain

$$A^2(s)\,n(s)\,d\sigma = A_0^2\,n_0\,d\sigma_0 \quad . \tag{4}$$

Here A, n and $d\sigma$ denote the amplitude, index of refraction and cross-sectional area of the tube at P, while A_0, n_0 and $d\sigma_0$ are the corresponding quantities at Q. From (4) we obtain

$$A(s) = A_0 \left[\frac{n_0}{n(s)}\ \frac{d\sigma_0}{d\sigma}\right]^{1/2} \quad . \tag{5}$$

The ratio $d\sigma_0/d\sigma$ is the limit of the area ratios as the tube shrinks to the ray.

By combining (3) and (5) we find for the field at P

$$u(P) = A_0 \left[\frac{n_0}{n(s)}\ \frac{d\sigma_0}{d\sigma}\right]^{1/2} \exp\left[ik\left\{\phi_0 + \int_0^s n[x(s)]\,ds\right\}\right] \quad . \tag{6}$$

In this equation $k = \omega/c$ is the propagation constant defined in terms of ω, the angular frequency of the field and c, the reference propagation speed used in defining n. A reference wavelength λ is related to k by the equation $k = 2\pi/\lambda$. The field (6) is associated with one ray through P. The total field at P is defined to be the sum of the fields associated with all the rays through P. We denote by u_i, u_r, u_t and u_d respectively the fields associated with incident rays (i.e., rays from the source), reflected rays, transmitted (refracted) rays and diffracted rays. We also call $u_i + u_r + u_t$ the geometrical optics field u_g. Thus the total field is

$$u(P) = u_g(P) + u_d(P) . \tag{7}$$

The prescription (6) for the calculation of the field must be supplemented to be used on reflected and refracted rays. On these rays we assume that the initial value of the field at the point of reflection Q is proportional to the incident field at Q. Thus $u_r(Q) = Ru_i(Q)$, $u_t(Q) = Tu_i(Q)$. The proportionality factors R and T are called the reflection and transmission coefficients. We assume that they depend upon the nature of the field, the local properties of the media at the point of reflection and the angle of incidence. Because they depend only upon local properties, they can be determined from the analysis of plane wave reflection and transmission by a plane interface. Once they are known, u_g can be calculated completely for any given u_i. As an example, if the reflecting surface is characterized by an impedance Z then

$$R = [n \cos \delta - Z][n \cos \delta + Z]^{-1} . \tag{8}$$

Here n and δ are the index of refraction and the angle of incidence at the point of reflection. This result can be obtained by inserting the expressions of the form (6) for u_i and u_r into the impedance boundary condition $\partial u/\partial \nu = ikZu$. In the limiting cases $Z = 0$ and $Z = \infty$, (8) yields $R = 1$ and $R = -1$ respectively.

Equation (6) is also applicable to diffracted rays with the exception of those portions of these rays which lie on the surface. Equation (3) for the phase is still applicable on these portions, but (5) must be modified. To modify it first denote by Q_1 the point of tangency of an incident ray with the surface. Then $u_d(Q_1)$ is the field on the diffracted (or surface) ray at Q_1. We now assume that the diffracted field at Q_1 is proportional to the incident field there

$$u_d(Q_1) = D(Q_1)u_i(Q_1). \tag{9}$$

The proportionality factor $D(Q_1)$ we call the diffraction coefficient. We assume that it depends upon the nature of the field, the properties of the media and the shape of the surface at Q_1, and the propagation constant k. In the case of vector fields D is a matrix. Before describing how D can be determined we will consider the variation of the diffracted field along the diffracted ray.

Since the phase of the diffracted field is given by (3), we need consider only the amplitude. According to (3), the phase of $u_d(Q_1)$ equals that of $u_i(Q_1)$. Therefore we have from (9)

$$A_d(Q_1) = D(Q_1)A_i(Q_1). \tag{10}$$

Let us denote by t the distance from Q_1 along the diffracted ray and by $A_d(t)$ the amplitude of the diffracted field at t. To determine $A_d(t)$ we will consider a narrow band of rays on the surface and apply to it the energy conservation principle. We first denote by $d\sigma(t)$ the width of the band. Then the energy flux along the band is proportional to $A_d^2 n d\sigma$. We assume that this flux diminishes due to radiation via the shedding of diffracted rays from the band. The energy removed by the shed rays must be proportional to the energy carried in the band so we assume that

$$\frac{d}{dt}\left[A_d^2\, n\, d\sigma\right] = -2\alpha(t)A_d^2\, n\, d\sigma. \tag{11}$$

The proportionality factor $2\alpha(t)$ depends upon local properties of the surface, the media and the field, in accordance with the localized nature of the radiation process. We will consider the determination of $\alpha(t)$ later.

Now we integrate (11) from $t = 0$ and obtain

$$A_d(t) = A_d(0)\left[\frac{n(0)d\sigma(0)}{n(t)d\sigma(t)}\right]^{1/2} \exp\left[-\int_0^t \alpha(t)dt\right] . \tag{12}$$

The ratio $d\sigma(0)/d\sigma(t)$ is the ratio of strip widths at 0 and t, or more precisely, the limit of this ratio as the strip width tends to zero. The point $t = 0$ is the point of tangency Q_1 so $A_d(0) \equiv A_d(Q_1)$. This initial value of A_d is given by (10). We now insert this into (12), include the phase given by (3) and denote $d\sigma(0)$ by $d\sigma(Q_1)$. Then (12) becomes

$$u_d(t) = D(Q_1)A_i(Q_1)\left[\frac{n(Q_1)}{n(t)}\ \frac{d\sigma(Q_1)}{d\sigma(t)}\right]^{1/2} \exp\left\{ik\left[\phi_i(Q_1)+\int_0^t n dt\right] - \int_0^t \alpha(t)dt\right\} . \tag{13}$$

Equation (13) gives the field on a diffracted ray on the surface.

Let us now determine the diffracted field $u_d(P)$ at a point P off the surface. First we denote by P_1 the point on the surface at which the diffracted ray through P leaves the surface. Then (13) gives $u_d(P_1)$. Next we attempt to employ (6) to calculate the field at P in terms of that at P_1. This cannot be done directly since the surface is a caustic of the diffracted rays and therefore in (6), $d\sigma_o$ is zero on the surface. Therefore, instead, we use (6) to relate $u_d(P)$ to $u_d(Q)$ where Q is any point near P_1 on the diffracted ray from P_1 to P. Then we let Q approach P_1. Since $d\sigma_o/d\sigma$ tends to zero, it follows from (6) that $A_o \equiv A_d(Q)$ must become infinite in such a way that $A_o(d\sigma_o/d\sigma)^{1/2}$ has a limit.

If s denotes the distance from Q to P_1 then $d\sigma_o \equiv d\sigma(Q)$ tends to zero like s. Therefore we may define $A_d'(P_1)$ and $d\sigma'(P_1)/d\sigma(P)$ by

(14) $$A_d'(P_1) = \lim_{Q \to P_1} s^{1/2} A_d(Q) .$$

(15) $$d\sigma'(P_1)/d\sigma(P) = \lim_{Q \to P_1} s^{-1} d\sigma(Q)/d\sigma(P).$$

With these definitions (6) becomes

(16) $$u_d(P) = A_d'(P_1)\left[\frac{n(P_1)}{n(P)} \frac{d\sigma'(P_1)}{d\sigma(P)}\right]^{1/2} \exp\left[ik\left\{\phi_d(P_1) + \int_0^s n[x(s)]ds\right\}\right] .$$

Since the diffracting surface is a caustic of the diffracted rays, $d\sigma(P)$ is zero when P is on the surface. Consequently (16) does not apply on this surface. In fact, the present theory always fails in this way at a caustic and requires a modification which will be described later. Therefore, we are unable to identify $A_d'(P_1)$ in (16) with the amplitude of the diffracted field on the surface at P_1, given in (13). Instead we assume that $A_d'(P_1)$ is proportional to this amplitude

(17) $$A_d'(P_1) = D(P_1)D(Q_1)A_i(Q_1)\left[\frac{n(Q_1)}{n(P_1)} \frac{d\sigma(Q_1)}{d\sigma(P_1)}\right]^{1/2} \exp\left[-\int_0^t \alpha(\tau)d\tau\right].$$

In (17), which determines $A_d'(P_1)$, the proportionality factor $D(P_1)$ is again called a diffraction coefficient. We assume that it is the same function of the properties of the field and object at P_1 as is $D(Q_1)$ of the properties at Q_1. This assumption is based upon the reciprocity principle - that a source at Q produces the same field at P as a source at P produces at Q. Our assumption assures that this principle is satisfied by the diffracted field. Now we insert (17) into (16) to obtain

(18) $$u_d(P) = D(P_1)D(Q_1)u_i(Q_1)\left[\frac{n(Q_1)}{n(P)} \frac{d\sigma(Q_1)}{d\sigma(P_1)} \frac{d\sigma'(P_1)}{d\sigma(P)}\right]^{1/2} \exp\left[\int_0^t (ikn-\alpha)dt + ik\int_0^s ndt\right] .$$

To obtain this result, we have necessarily excluded cases in which the field or some component of it has to vanish on the boundary. This was required because our deductions were based on the amplitude function $A_d(t)$ of the field on the surface. As previously noted, the surface is a caustic of the diffracted field. Therefore the field must be much stronger within a surface layer a few wavelengths thick than at points a great number of wavelengths from the caustic. Thus the argument used to derive (18) still applies if $A_d(t)$ is any measure of the amplitude in this layer. However, the argument will now cover the cases which were previously excluded. Within the layer, the field will be characterized by some definite variation with distance from the surface. At any point other than a zero of this profile, the field amplitude can serve as the measure of the field in this layer.

The result in (18) does not account for the fact that, generally, the field within the caustic layer will be composed of different modes, each with its own profile. Each mode will then have its respective amplitude A_{dm} and thus its own diffraction coefficient $D_m(Q_1)$ and proportionality factor $\alpha_m(t)$. Therefore, (18) must be replaced by

$$(19) \qquad u_d(P) = u_i(Q_1)\left[\frac{n(Q_1)}{n(P)} \cdot \frac{d\sigma(Q_1)}{d\sigma(P_1)} \cdot \frac{d\sigma'(P_1)}{d\sigma(P)}\right]^{1/2} \exp\left[ik\int_0^t n\,dt + ik\int_0^s n\,ds\right]$$

$$\cdot \sum_m D_m(P_1) D_m(Q_1) \exp\left[-\int_0^t \alpha_m(\tau)\,d\tau\right].$$

The diffraction coefficients D_m and the decay exponents α_m must now be determined. These coefficients have previously been studied in cases of diffraction by convex objects in homogeneous media [2],[4]. In those cases it was found that the only geometrical property of the surface upon which D_m and α_m depend is the normal curvature of the surface in a plane tangent to the diffracted ray. However, in those cases the rays not lying on the surface are straight lines whereas in the present problems they are curved. The physical nature of the diffraction process suggests that in the present case D_m and α_m will depend only upon the relative curvature of the incident or diffracted ray and the surface. By 'relative curvature' we mean the sum of the curvatures of the ray and the normal section of the surface in a plane containing the incident or diffracted ray. Furthermore we assume that D_m and α_m are the same functions of this 'relative curvature' as they are of the surface curvature in the homogeneous case. Of course the value of the index of refraction n which appears in the expressions for D_m and α_m must be that at the point at which D_m or α_m is to be evaluated.

Let us consider a scalar field satisfying the reduced wave equation in an inhomogeneous medium and an impedance boundary condition on the body. From the known values of D_m and α_m for such a field in a homogeneous medium these coefficients can be evaluated for an inhomogeneous medium by means of the foregoing assumption. In [2],[4] the coefficients were found for a homogeneous medium in which $n = 1$. To obtain the corresponding results for a homogeneous medium with any other value of n it is merely necessary to replace k by kn in those results. Then from the above assumptions we obtain

$$(20) \quad D_m(P_1) = \frac{\pi^{3/4}\left[kn(P_1)\right]^{-1/12}\left[K(P_1)+b^{-1}(P_1)\right]^{-1/6}\exp\left[\frac{i\pi}{24} - 2i(3^{-1}q_m)^{3/2}\right]}{6^{1/6}2^{1/4}\left[q_m A^2(q_m) + 3A'^2(q_m)\right]^{1/2}}$$

$$(21) \quad \alpha_m(P_1) = e^{-i\pi/6}\, q_m\left[6^{-1}kn(P_1)\right]^{1/3}\left[K(P_1)+b^{-1}(P_1)\right]^{2/3}.$$

In these equations $K(P_1)$ and $b^{-1}(P_1)$ are respectively the curvature of the ray and of the normal section of the surface at P_1. $A(q_m)$ is the Airy function given by (53), Part II and q_m is the m-th zero of (52), Part II.

4. Plane waves in a plane stratified medium

As a first application of the general method just described, we consider an unbounded, plane-stratified medium. The index of refraction is $n = n(x)$ and tends to the constants n_o, n_1 respectively as x approaches $\pm\infty$. We suppose that the incident field at $x = -\infty$ is

$$(22) \qquad u_i = e^{ikn_o(x \cos \alpha + y \sin \alpha)} .$$

For this medium, (1) reduces to

$$(23) \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2} n^2\right).$$

A first quadrature of (22) yields

$$(24) \qquad \frac{dy}{dx} = \pm \frac{a}{(n^2-a^2)^{1/2}} .$$

By choosing the positive sign and letting x tend to $-\infty$, we deduce that $a = n_o \sin \alpha$. The multivaluedness of the slope suggests that more than one ray can pass through any point. The number of rays depends on the number of zeros x_α of the equation

$$(25) \qquad n^2(x_\alpha) - a^2 = 0.$$

We call points at which (25) is satisfied turning points since at them the rays become vertical and are turned back toward $-\infty$.

We will consider three cases. If $n_o \sin \alpha$ is less than the minimum of $n(x)$, there are no turning points. The rays then form a family of congruent curves proceeding always to the right. If $n_o \sin \alpha$ lies between the minimum and maximum of $n(x)$, then (25) has at least one solution. If there is only one solution (turning point) $x = x_\alpha$, all the rays become vertical along the line $x=x_\alpha$

and are refracted back toward $x = -\infty$. The region $x > x_\alpha$ is then a refraction shadow and is void of real rays. We will also consider the case when there is an additional turning point at $x = x_\alpha' > x_\alpha$.

(a) No turning points

In this case, we immediately obtain the equations of all the rays by integrating (24), choosing the positive sign. We find

$$(26) \qquad y = \int_{x_0}^{x} \frac{a\,dx}{(n^2-a^2)^{1/2}} + y_0.$$

If we use the value of a in terms of α, this can be put into the form

$$(27) \qquad y = x \tan\alpha + \int_{-\infty}^{x} \left[\frac{n_0 \sin\alpha}{(n^2-n_0^2 \sin^2\alpha)^{1/2}} - \tan\alpha \right] dx + b.$$

Thus, for large negative x, the rays behave like straight lines.

To find the phase along these rays, we first transform (3) to the parameter t using (2) to show that $ds/dt = n$. Then we use the value of dx/dt obtained by integrating the second of (23), and we find

$$(28) \qquad \phi = \phi_0 + a(y-y_0) + \int_{x_0}^{x} (n^2-a^2)^{1/2} dx.$$

The constant ϕ_0 can be determined by comparing (28) with u_i at $x = -\infty$. Then (28) can be written in the form

$$(29) \qquad \phi = n_0(x \cos\alpha + y \sin\alpha) + \int_{-\infty}^{x} g(x) dx.$$

Here, $g(x)$ is defined by

$$g(x) = (n^2-n_o^2\sin^2\alpha)^{1/2} - n_o\cos\alpha. \tag{30}$$

Now we find the amplitude by applying (5). By noting that $d\sigma/dy=\left[1+(\frac{dy}{dx})^2\right]^{-1/2}$, we find that

$$\frac{d\sigma_o}{d\sigma} = \frac{d\sigma_o}{dy}\Big/\frac{d\sigma}{dy} = \frac{n}{n(x_o)}\left[\frac{n^2(x_o) - n_o^2\sin^2\alpha}{n^2-n_o^2\sin^2\alpha}\right]^{1/2}. \tag{31}$$

Upon using (31) in (5) and letting $x_o \to -\infty$, we obtain

$$A = A(x_o)\left[\frac{n^2(x_o)-n_o^2\sin^2\alpha}{n^2-n_o^2\sin^2\alpha}\right]^{1/4} = \frac{A_o(n_o\cos\alpha)^{1/2}}{(n^2-n_o^2\sin^2\alpha)^{1/4}}. \tag{32}$$

Comparison of (32) with u_i at $-\infty$ shows that $A_o = 1$. Finally, by combining (28) and (32) we obtain for the field

$$u = \frac{(n_o\cos\alpha)^{1/2}}{(n^2-n_o^2\sin^2\alpha)^{1/4}}\exp\left[ikn_o(x\cos\alpha + y\sin\alpha) + ik\int_{-\infty}^{x} g(x)dx\right] \tag{33}$$

This result agrees exactly with (6) of Part II, which was obtained by asymptotically expanding the solution of the corresponding boundary value problem.

(b) One turning point

Let the turning point be at $x = x_\alpha$. The incident rays therefore become vertical at $x = x_\alpha$ and the line $x = x_\alpha$ is then a caustic. At the plane $x = x_\alpha$, the incident ray splits into a real ray refracted back towards $x = -\infty$ and a complex ray (see Figure 2). This corresponds to $(n^2- a^2)^{1/2}$ being pure imaginary for $x > x_\alpha$ so that there are no real solutions of the ray equation. The region $x > x_\alpha$ is a refraction shadow in which there are only complex or imaginary rays. A complex or imaginary ray is a complex-valued solution of the ray equation. We say that it lies in a given region if the real part of x lies in that region. The field 'along' such a ray is found by utilizing the analytic continuations of

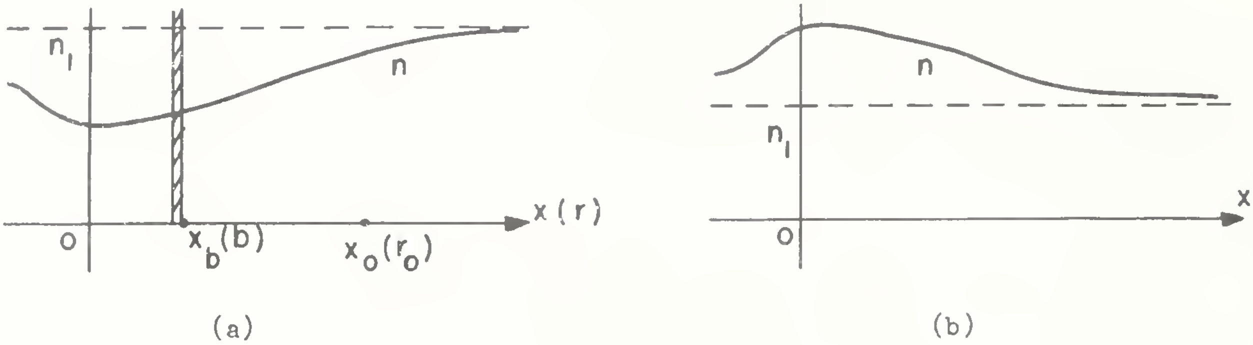

Figure 1. Typical graphs of the index of refraction $n(x)$ or $n(r)$ in a plane or cylindrically stratified medium. n tends to n_1 at infinity. If the index is of the form in (b) the medium will contain a duct; if it is of the form in (a) there will be no duct.

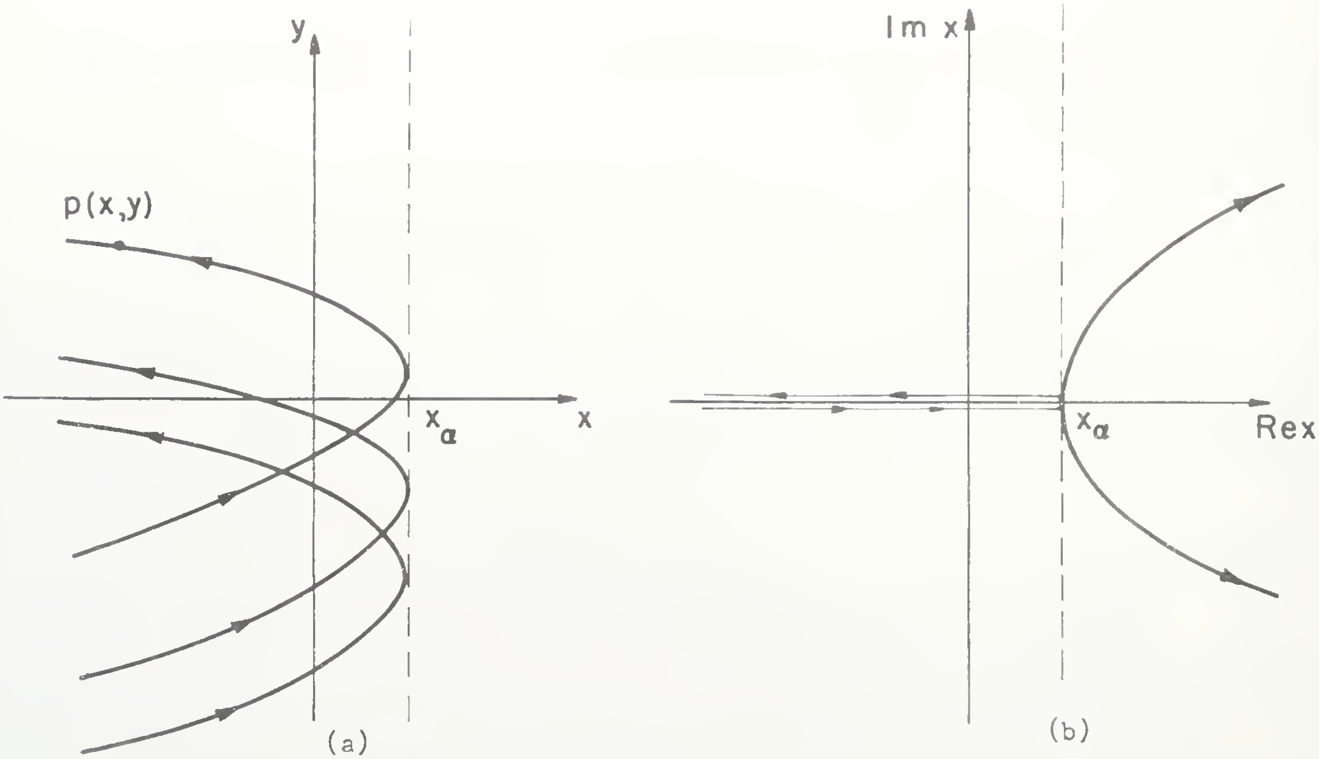

Figure 2. (a) The ray pattern in a plane stratified medium due to a plane wave incident from the left. There is one turning point at x_α. The incident rays have the slope $\tan \alpha$ at $-\infty$. (b) The ray pattern in the complex x-plane. The real rays become vertical at $x = x_\alpha$ and then turn back to the left. The region $x > x_\alpha$ is a refraction shadow and only imaginary or complex rays occur here. For Re $x < x_\alpha$, there are real rays.

(3)-(6). The equation of a complex ray is the analytic continuation of the incident ray.

The incident rays and the phase and amplitude associated with them are again given by (26), (28) and (32) respectively. In each, however, we replace (x_o,y_o) by (x_α,y_α) and all three formulas hold only for $x \leq x_\alpha$.

The equation of a refracted ray after turning is obtained by reflecting the incident ray across the horizontal line $y = y_\alpha$ through the turning point (x_α,y_α). The ray is given by (26) with a minus sign in front of the integral and (x_α,y_α) instead of (x_o,y_o). This result can be put into a form similar to (27). This form again shows that a refracted ray behaves, as x tends to $-\infty$, like the straight line with slope $-\tan\alpha$.

The phase along a refracted ray after turning is obtained by using (3) with ϕ_o the phase of the incident ray at $x = x_\alpha$. We thus obtain

$$(34) \qquad \phi = n_o(-x\cos\alpha + y\sin\alpha) - \int_{-\infty}^{x} g(x)dx + 2\left(n_o x_\alpha \cos\alpha + \int_{-\infty}^{x_\alpha} g(x)dx\right) .$$

We recall that $x = x_\alpha$ is a caustic for the real rays. When a ray (in a homogeneous medium) passes through a caustic, the phase of the field is retarded by $\pi/2$[16]. The field along an incident ray here has the same local behavior at $x = x_\alpha$. With this phase change taken into account, the amplitude along a refracted ray is given by

$$(35) \qquad A = \frac{(n_o\cos\alpha)^{1/2}e^{-\pi i/2}}{(n^2-n_o^2\sin^2\alpha)^{1/4}} .$$

The above results, when collected, yield the field at any point.

$$(36)\quad u = \frac{(n_o \cos \alpha)^{1/2}}{(n^2 - n_o^2 \sin^2 \alpha)^{1/4}} \Bigg\{ \exp\left[ikn_o(x \cos \alpha + y \sin \alpha) + ik \int_{-\infty}^{x} g(x)dx \right]$$

$$+ \exp\left[ikn_o(-x \cos \alpha + y \sin \alpha) - ik \int_{-\infty}^{x} g(x)dx + 2ik \int_{-\infty}^{x_\alpha} g(x)dx + 2ikn_o x_\alpha \cos \alpha - i\pi/4 \right] \Bigg\}, \quad x < x_\alpha .$$

This result agrees exactly with (11), Part II obtained by asymptotically expanding the solution of the corresponding boundary value problem.

In the refraction shadow, the equation of a ray, the phase and the amplitude are given by the analytical continuations of the corresponding formulas (26), (28) and (32) for an incident ray. The field at any point (x,y) in this region is given by

$$(36')\quad u = \frac{(n_o \cos \alpha)^{1/2}}{(n_o^2 \sin^2 \alpha - n^2)^{1/4}} \left[ikn_o y \sin \alpha - k\;(x) + ik \int_{-\infty}^{x_\alpha} g(x)dx + ikn_o x_\alpha \cos \alpha - i\pi/4 \right], \quad x > x_\alpha .$$

This agrees with (12), Part II, the asymptotic expansion of the exact solution to the corresponding boundary value problem.

(c) Two turning points

Let the two turning points be at x_α and x_α', $x_\alpha < x_\alpha'$. The lines $x = x_\alpha$, $x = x_\alpha'$ are caustics for the real rays in the regions $x \leq x_\alpha$, $x \geq x_\alpha'$ respectively. We will label these half-spaces I and III. The region $x > x_\alpha$ is a refraction shadow and in the region between the two caustics (which we label II) there are no real rays. However, we will now show that through every point in the space there pass an infinite number of rays. To see this we note that at the plane $x = x_\alpha$ the incident ray splits into a ray refracted back into I and an imaginary or complex ray continuing into II. At the plane $x = x_\alpha'$, this imaginary ray splits into an imaginary ray refracted back and a real ray continuing into III. This splitting occurs each time a complex ray reaches either of the planes $x = x_\alpha$ or $x = x_\alpha'$. Each of the additional real rays which arise because of the splitting of a complex ray is coincident with one of the initial real refracted or transmitted rays. Consequently there will be infinitely many real rays passing through each point in I and III and infinitely many complex rays through each point in II (see Figure 3,4).

In I, the equations of an incident ray and its continuation beyond the turning point are the same as those equations given in the section on the one-turning point case and in the region $x < x_\alpha$. The equations of the complex rays in II are the appropriate analytic continuations. The rays in III are given by (26) with (x_o,y_o) replaced by (x_α',y_α').

Now let $\phi_n^t(P)$ and $\phi_n^r(P)$ stand for the phase at a point P along a ray. The superscripts t and r mean that the ray at that point is being transmitted to the right or to the left respectively. The subscripts I, II, or III refer to the particular region in which P lies. The subscript n indicates the number of times any one of the complex rays described above(which leads analytically to a ray being considered)has 'reflected' at $x = x_\alpha$ and $x = x_\alpha'$. Then ${}_{I}\phi_0^t$ and ${}_{I}\phi_0^r$ are given respectively by (29) and (34). ${}_{II}\phi_0^t$ is given by the analytic continuation of ${}_{I}\phi_0^t$. In general, we have for $m = 1, 2, \ldots$

$$(37) \qquad {}_{I}\phi_{2m}^r = n_o(y \sin\alpha + x_\alpha \cos\alpha) - \int_{x_\alpha}^{x} (n^2 - n_o^2 \sin^2\alpha)^{1/2} dx + \int_{-\infty}^{x_\alpha} g(x)dx$$

$$+ 2mi \int_{x_\alpha}^{x_\alpha'} (n_o^2 \sin^2\alpha - n^2)^{1/2}\, dx,$$

$$(38) \qquad {}_{II}\phi_{2m}^t = n_o(y \sin\alpha + x_\alpha \cos\alpha) + i \int_{x_\alpha}^{x} (n_o^2 \sin^2\alpha - n^2)^{1/2} dx + \int_{-\infty}^{x_\alpha} g(x)dx$$

$$+ 2mi \int_{x_\alpha}^{x_\alpha'} (n_o^2 \sin^2\alpha - n^2)^{1/2}\, dx$$

(39) $${}_{II}\phi^{r}_{2m-1} = n_o(y \sin\alpha + x_\alpha \cos\alpha) + i\int_{x}^{x'_\alpha} (n_o^2 \sin^2\alpha - n^2)^{1/2} dx + \int_{-\infty}^{x_\alpha} g(x)dx$$
$$+ i(sm-1)\int_{x_\alpha}^{x'_\alpha} (n_o^2 \sin^2\alpha - n^2)^{1/2} dx$$

(40) $${}_{III}\phi^{t}_{2m-1} = n_o(y \sin\alpha + x_\alpha \cos\alpha) + \int_{x'_\alpha}^{x} (n^2 - n_o^2 \sin^2\alpha)^{1/2} dx + \int_{-\infty}^{x_\alpha} g(x)dx$$
$$+ i(2m-1)\int_{x_\alpha}^{x'_\alpha} (n_o^2 \sin^2\alpha - n^2)^{1/2} dx.$$

We now use (5) again to construct the amplitude, making the same assumption as for the one-turning point case and an additional assumption. Namely, when a complex ray splits, the amplitude along the resulting complex ray is initially multiplied by $\frac{1}{2} e^{\pi i/2}$*. Therefore, if we define A^t_n, A^r_n in a similar way, we find that ${}_{I}A^t_o$ and ${}_{I}A^r_o$ are given respectively by (32) and (35) and that ${}_{II}A^t_o$ is given by the analytic continuation of (32). In general, we find for m = 1, 2, ...

(41) $${}_{I}A^r_{2m} = 2^{1-2m}\ {}_{I}A^t_o\ e^{\frac{2m-1}{2} i\pi}$$

(42) $${}_{II}A^t_{2m} = 2^{-2m}\ {}_{II}A^t_o\ e^{m\pi i}$$

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*To justify this, we consider the canonical problem: To find the field which is outgoing as x tends to $-\infty$ in a medium with an index of Figure 1a (with only one turning point at $x = x_\alpha$). The method of solution parallels that for the one-turning point case and yields the amplitude change described above.

$$(43)\qquad {}_{II}A^{r}_{2m-1} = 2^{1-2m}\;{}_{II}A^{t}_{o}\,e^{\frac{2m-1}{2}\pi i}$$

$$(44)\qquad {}_{III}A^{t}_{2m-1} = 2^{2-2m}\;{}_{I}A^{t}_{o}\,e^{(m-1)\pi i}\;.$$

The contributions at an arbitrary point in I, II or III form geometric series and when summed yield the fields in the respective regions. We find that in region I

$$(45)\qquad u = \frac{(n_o \cos\alpha)^{1/2}}{(n^2 - n_o^2\sin^2\alpha)^{1/4}}\left\{\exp\left[ikn_o(x\cos\alpha + y\sin\alpha) + ik\int_{-\infty}^{x} g(x)dx\right]\right.$$

$$+\frac{4 - e^{-2k\eta(x'_\alpha)}}{4 + e^{-2k\eta(x'_\alpha)}}\exp\left[ikn_o(-x\cos\alpha + y\sin\alpha) - ik\int_{-\infty}^{x} g(x)dx\right.$$

$$\left.\left. + 2ik\int_{-\infty}^{x_\alpha} g(x)dx + 2ikn_o x_\alpha\cos\alpha - \frac{i\pi}{2}\right]\right\}, \qquad x < x_\alpha\;.$$

Here, $\eta(x)$ is defined by

$$(46)\qquad \eta(x) = \int_{x_\alpha}^{x}(n_o^2\sin^2\alpha - n^2)^{1/2}\,dx.$$

In the interval between the turning points, the field is

$$(47)\qquad u = \frac{2(n_o\cos\alpha)^{1/2}\,e^{k\eta(x'_\alpha)}}{(n_o^2\sin^2\alpha - n^2)^{1/4}(1+4e^{2k\eta(x'_\alpha)})}\exp\left[ikn_o(y\sin\alpha + x_\alpha\cos\alpha) + ik\int_{-\infty}^{x_\alpha} g(x)dx\right]$$

$$\left\{\exp\left[k\eta(x) + \frac{i\pi}{4}\right] + 2\exp\left[-k\eta(x) - \frac{i\pi}{4}\right]\right\}.$$

When $x > x'_\alpha$, we find

$$(48)\qquad u = \frac{4(n_o \cos\alpha)^{1/2}\, e^{k\eta(x'_\alpha)}}{(n_o^2\sin^2\alpha - n^2)^{1/4}\,(1+4e^{2k\eta(x'_\alpha)})}\exp\Big[ikn_o(y\sin\alpha + x_\alpha\cos\alpha)$$

$$+ ik\int_{-\infty}^{x} g(x)dx + ik\int_{x'_\alpha}^{x}(n^2-n_o^2\sin^2\alpha)^{1/2}\,dx\Big].$$

These results coincide with (11), (13) and (14) of Part II. The latter expressions are obtained by expanding the exact solutions asymptotically.

The preceding construction has been done in a heuristic way. By looking at the complex-valued solutions of (24) we can obtain a more precise description of the rays and associated fields. The integrand of (26) is actually a double-valued function and may be thought of as being defined on a two-sheeted Riemann surface in the complex x-plane with one branch point occurring at $x = x_\alpha$. This corresponds to the case of one turning point where we assume that n^2-a^2 has a simple zero at $x = x_\alpha$. It can be shown that the integral around any circle in the x-plane with x_α as center approaches zero as the radius goes to zero. Thus it makes sense to talk about integrating up to x_α and then from x_α (perhaps on the other sheet) to some other point. Thus y is a double-valued function of x everywhere except at $x = x_\alpha$. Since only real values of x are of interest, the paths of integration will be confined to the real axis but on two sheets. With this interpretation, (26) with (x_o,y_o) replaced by (x_α,y_α) clearly gives the two branches of the 'real' ray corresponding to the incident and refracted ray for $x \leq x_\alpha$. That is, on the upper sheet we take (34) and on the lower sheet we take (34) with a negative sign.

The Riemann surface associated with the integrand of the phase on a ray is the same as that just described. This may be seen by substituting in (28) the value of y from the equation of the ray. Therefore, the determination of the radical for ϕ corresponds to the determination of y.

According to (32) the amplitude is a four-valued function of x. As we go around x_α in the direction in which arg $(n^2-n_o^2\sin^2\alpha)$ increases we can follow the determination of the argument of A modulo π. This indeterminacy is resolved by comparing the field construction with the asymptotic expansion of the exact solution.

When there are two turning points at x_α, x_α', the Riemann surfaces of the integrands of (26) and (28) are again two-sheeted. However, because of the two branch points, the functions y and ϕ are infinite-valued. At any point P, the phase and amplitude will be given by (37)-(40) and (41)-(44) respectively. A particular value of m merely corresponds to the sheet on which the path of integration to P is chosen.

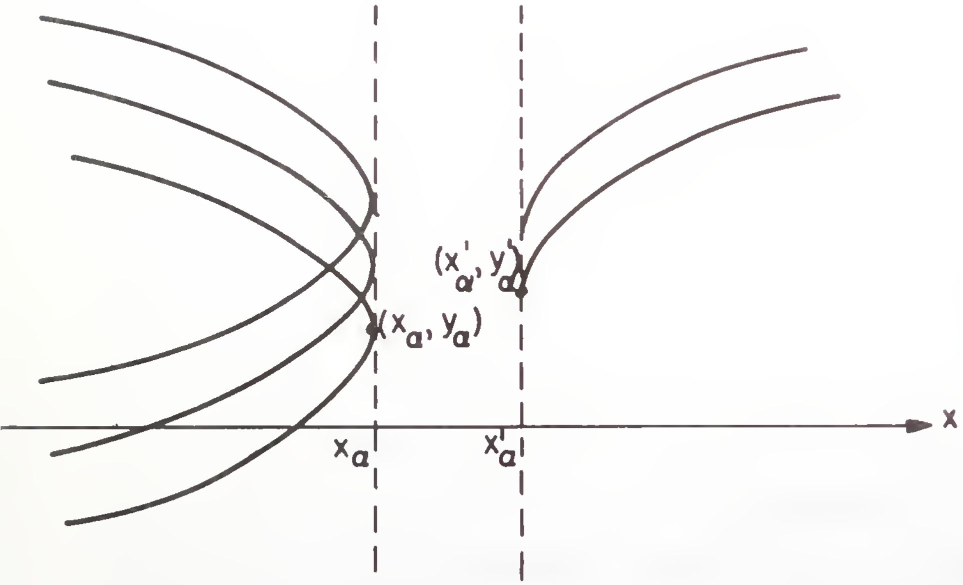

Figure 3

The ray pattern in a plane stratified medium due to a plane wave incident from the left. The incident rays make the angle α with the x-axis at $-\infty$. The rays become vertical at $x=x_\alpha$ and then turn back to the left. Only imaginary or complex rays occur in the interval from x_α to x_α', and at x_α' they give rise to real transmitted rays. The values x_α and x_α' are the two roots of the equation $n^2(x)-n_o^2\sin^2\alpha=0$ where n_o is the value of n(x) as x tends to $-\infty$.

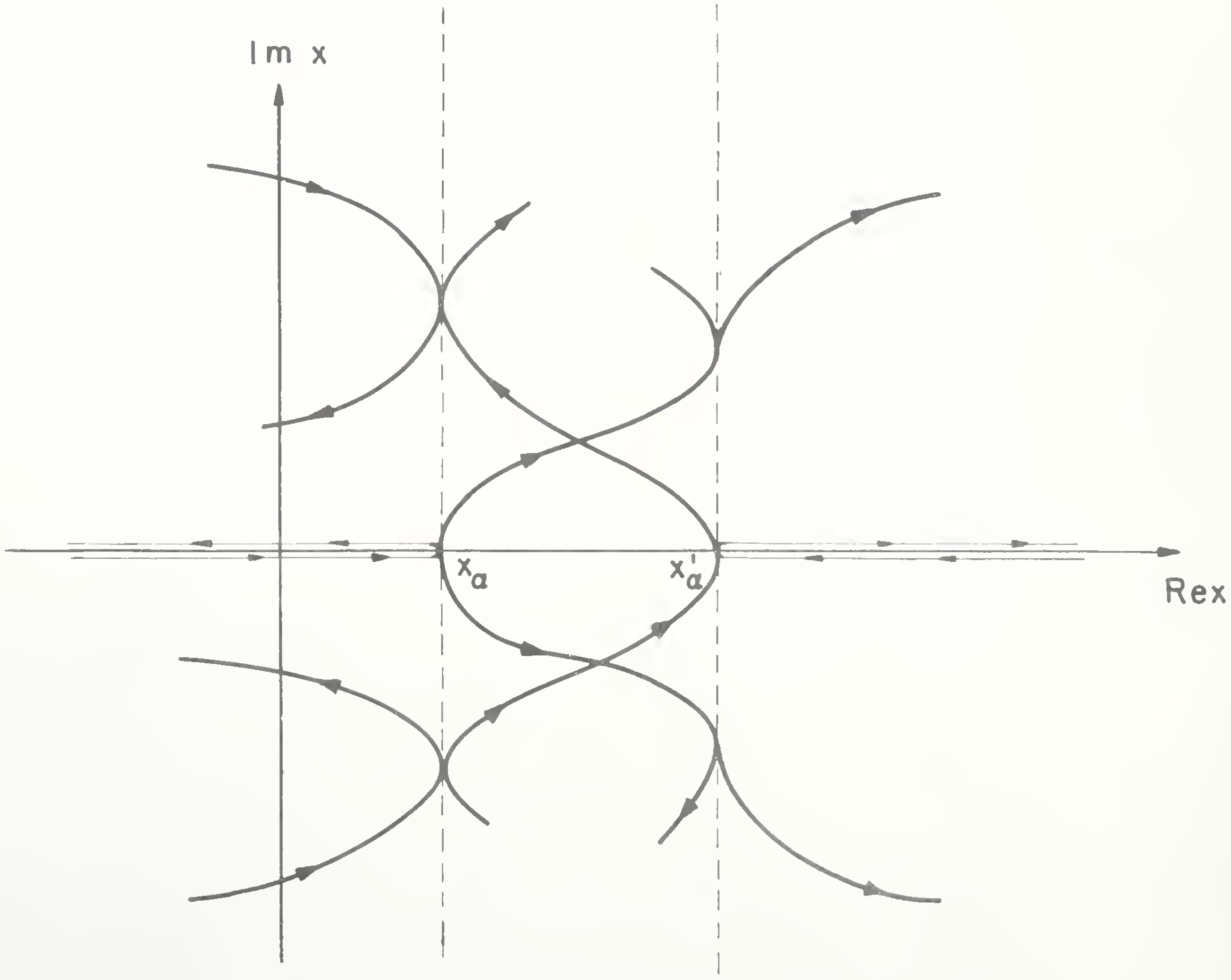

Figure 4

The ray pattern in the complex x-plane in a plane stratified medium corresponding to the pattern of Figure 3. The lines along the real axis are the real rays and the curves the complex rays.

5. The field of a line source in a plane stratified medium with a plane boundary

In this section we construct the field due to a line source parallel to the z-axis and located at $(x_0,0)$. The index of refraction $n = n(x)$ tends to n_1 as x approaches infinity and satisfies the condition $n(x) > n(x_0)$ for $x > x_0$. The medium is bounded on the left by the plane $x = x_b$ on which the field satisfies the impedance condition

$$\frac{\partial u}{\partial x} = ikZu, \qquad x = x_b < x_0. \tag{49}$$

In this medium, the rays are determined by (24). The constant a in (24) is given by $a = n(x_0) \sin \alpha$ where $\tan \alpha$ is the slope of a ray at the source. Those rays for which $0 \leq \alpha \leq \pi/2$ proceed to the right without turning. They are given by (26) with $y_0 = 0$. Of the rays for which $\pi/2 \leq \alpha \leq \pi$ some hit the boundary and are reflected, while others are turned back to the right before reaching the boundary. We will call a ray from the source a direct or incident ray until it is reflected or turned; then we will call it a reflected or a refracted ray. For one particular value of α, say $\alpha = \alpha_b$, the ray grazes the bounding plane. The continuation of this ray is the boundary of a refraction shadow. The plane $x = x_b$ is the other boundary of the shadow. In the shadow there are neither incident, reflected nor refracted rays but there are diffracted rays (see Figure 5). All the rays in the range $\alpha_b < \alpha \leq \pi$ are reflected while those in the range $\pi/2 < \alpha < \alpha_b$ are refracted.

To calculate the phase along the incident rays we apply (3) and obtain the result (28) with y_0 and ϕ_0 both zero. To determine the amplitude, we apply (5) and obtain an expression of the form (32) with the numerator and denominator respectively multiplied by $\partial y_0/\partial a$ and $\partial y/\partial a$. Calculating these derivatives from the equations of the rays and denoting by A_0 all the constant terms that arise, we find that

$$(51) \qquad A = A_o\left[n^2-a^2\right]^{-1/4}\left[\int_{x_<}^{x_>}\frac{n^2dx}{(n^2-a^2)^{3/2}}\right]^{-1/2} .$$

Here $x_>$ and $x_<$ are respectively the larger and smaller of x and x_o. The constant A_o depends upon the strength of the source. If the line source is of unit strength*, then

$$(52) \qquad A_o = \left[\frac{i}{8\pi k}\right]^{1/2}\left[n^2(x_o) - a^2\right]^{-1/4} .$$

By combining (51) and (52) with the phase formula, we finally obtain for the field on an incident ray

$$(53) \qquad u_i = \left[n^2(x_o)-a^2\right]^{-1/4}\left[n^2-a^2\right]^{-1/4}\left[8\pi k\int_{x_<}^{x_>}\frac{n^2dx}{(n^2-a^2)^{3/2}}\right]^{-1/2}$$

$$\cdot \exp\left[ik\int_{x_<}^{x_>}(n^2-a^2)^{1/2}dx + ikay + \frac{i\pi}{4}\right] .$$

Each ray for which $\alpha_b < \alpha \leq \pi$ meets the boundary. The resulting reflected ray can be obtained by reflecting the incident ray across the horizontal line y = constant = $y_i(x_b)$ through the point of reflection. The reflected ray is therefore given by

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*By a line source of unit strength at $(x_o,0)$, we mean one near which the field behaves like the Hankel function $i/4\ H_0^{(1)}\left[kn(x_o)r\right]$ where r denotes distance from the source.

$$(54) \qquad y = 2y_i(x_b) - y_i = \left\{ \int_{x_b}^{x} + \int_{x_b}^{x_o} \right\} \frac{a\,dx}{(n^2-a^2)^{1/2}}, \qquad x_b \le x < \infty.$$

The field along the reflected ray can now be obtained by using (3), (5), (8) and the relation $u_r(Q) = Ru_i(Q)$. The value of $d\sigma_o/d\sigma$ in (5) is obtained as in the case of the incident rays, $\partial y_o/\partial a$ and $\partial y/\partial a$ being calculated from (54). The quantity $\cos\delta$ can be determined from the equation of the incident ray and we find that

$$(55) \qquad \cos\delta = -\left[n^2(x_b) - a^2\right]^{1/2}/n(x_b).$$

Combining these results with the value of u_i at the point of reflection, which can be obtained from (53), we find

$$(56) \quad u_r = R\left[n^2(x_o)-a^2\right]^{-1/4}\left[n^2-a^2\right]^{-1/4}\left[8\pi k\left\{\int_{x_b}^{x} + \int_{x_b}^{x_o}\right\}\frac{n^2dx}{(n^2-a^2)^{3/2}}\right]^{-1/2}$$

$$\cdot \exp\left[ik\left\{\int_{x_b}^{x} + \int_{x_b}^{x_o}\right\}(n^2-a^2)^{1/2}dx + ikay + \frac{i\pi}{4}\right].$$

The reflection coefficient R is given by

$$(57) \qquad R = \frac{\left[n^2(x_b) - a^2\right]^{1/2} + Z}{\left[n^2(x_b) - a^2\right]^{1/2} - Z}.$$

The expressions in (53) and (57) agree exactly with (38) and (39) of Part II, which were obtained by asymptotically expanding the solution of the corresponding boundary value problem.

Those rays corresponding to the parameter range $\frac{\pi}{2} \le \alpha < \alpha_b$ turn back before meeting the boundary. The equations of these rays and the field along them up to their turning points are determined by the preceding formulas which are now valid in the interval $x_a \le x \le x_o$. We note that (51) and (53) are indeterminate at $x = x_a$. If we evaluate the limit of (51) as x tends to x_a we find for A at a turning point the result

$$(58) \qquad A = \left[\frac{iK(x_a)}{8\pi k}\right]^{1/2} \left[n^2(x_o) - a^2\right]^{-1/4}.$$

Here $K(x_a) = n'(x_a)/n(x_a)$ is the curvature of the ray at the turning point. Thus the field at a turning point is given by an expression of the form (53) with (58) used for A instead of (51). This result again coincides with the asymptotic expansion of the exact solution of the boundary value problem given in Part II.

Beyond the turning point of a refracted ray, the equation of the ray, is given by (54) with x_b replaced by x_a in the lower limits of the integrals as we see from the symmetry of the ray. The same modification yields the phase. To find the amplitude, we use (5). As before, in simplifying this expression, we have to calculate $\partial y_o/\partial a$ and $\partial y/\partial a$ from the equations of the rays. Differentiation of (54) cannot be performed directly since the resulting boundary term and integral are both infinite. One way in which this difficulty can be avoided is to use the smoothness of $n(x)$ to integrate (54) by parts. The resulting expression can then be differentiated to obtain the required derivatives.

Now we consider the shadow region. The diffracted rays which cover this region are produced by the single incident ray for which $\alpha = \alpha_b$. This ray is tangent to the bounding plane at the point $(x_b, Y_b) = Q_1$. The resulting surface ray proceeds along the boundary $x = x_b$ as a straight line. At each point it sheds a diffracted ray so all the diffracted rays of (24) are given by the one parameter family of congruent curves

$$(59) \qquad y = \int_{x_b}^{x} \frac{n(x_b)\,dx}{\left[n^2-n^2(x_b)\right]^{1/2}} + y_b .$$

Here, $(x_b,y_b) = P_1$ is the point of shedding of the rays through $(x,y) = P$.

To find the field associated with these rays, we now utilize (14)-(21). The surface rays are parallel lines so that on the surface $d\sigma_0/d\sigma = 1$. On the other hand, because the diffracted rays are congruent, we find on evaluating the limit in (15), that

$$(60) \qquad A_d(P) = A_d'(P_1)\left[n^2-n^2(x_b)\right]^{-1/4}\left[n(P_1)\,K\,(P_1)\right]^{1/2}.$$

Since the diffracting surface is a plane on which $n(x)$ is constant, the diffraction coefficients at P_1 and Q_1 are equal and $\alpha_m(\tau)$ is constant. Furthermore all other quantities in (60) may be evaluated at Q_1, the point of tangency of the limiting ray, instead of at P_1. By combining (60) with (17), we find for $A_d(P)$ the result

$$(61) \qquad A_d(P) = A_i(Q_1)\left[n(Q_1)\,K\,(Q_1)\right]^{1/2} D_m^2\left[n^2-n^2(x_b)\right]^{-1/4} e^{-\alpha_m(y_b-Y_b)} .$$

To complete the calculation of u_d we note that the phase change along the surface ray is just

$$(62) \qquad n(x_b)\,(y_b - Y_b).$$

Furthermore, $A_i(Q_1)$ can be determined from (58). The phase along the diffracted rays can be obtained from (5) by using (62) and the phase of the incident field at Q_1. In this way we find

$$(63) \qquad \phi(P) = n(x_b)y + \left\{ \int_{x_b}^{x} + \int_{x_b}^{x_o} \right\} \left[n^2 - n^2(x_b) \right]^{1/2} dx.$$

If for any point in the refraction shadow $y < 0$, the calculation of the ray equation through P and the field at P can be carried out in the same manner.

Collecting these results, we finally obtain the following expression for the diffracted field

$$(64) \qquad u_d = \sum_m D_m^2 \left[n^2 - n^2(x_b) \right]^{-1/4} \left[n^2(x_o) - n^2(x_b) \right]^{-1/4} K(x_b) \left[\frac{n(x_b)}{8\pi k} \right]^{1/2}$$

$$\exp\left[ik \left\{ \int_{x_b}^{x} + \int_{x_b}^{x_o} \right\} \left[n^2 - n^2(x_b) \right]^{1/2} dx + ik\, n(x_b)|y| - \alpha_m |y_b - Y_b| + \frac{\pi i}{4} \right].$$

The diffraction coefficient D_m and the decay exponent α_m are obtained from (20) and (21) by letting $b \to \infty$. This yields

$$(65) \qquad D_m = \frac{\pi^{3/4} \left[kn(x_b) \right]^{-1/12} K^{-1/6}(x_b) \exp\left[\frac{i\pi}{24} - 2i(3^{-1} q_m)^{3/2} \right]}{6^{1/6}\, 2^{1/4} \left[q_m A^2(q_m) + 3A'^2(q_m) \right]^{1/2}},$$

$$(66) \qquad \alpha_m = e^{-i\pi/6} q_m \left[6^{-1} kn(x_b) \right]^{1/3} K^{2/3}(x_b).$$

The result (64) with these values for D_m and α_m agrees exactly with the asymptotic expansion of the diffracted field obtained from the exact solution of the corresponding boundary value problem in Part II, and given by (56).

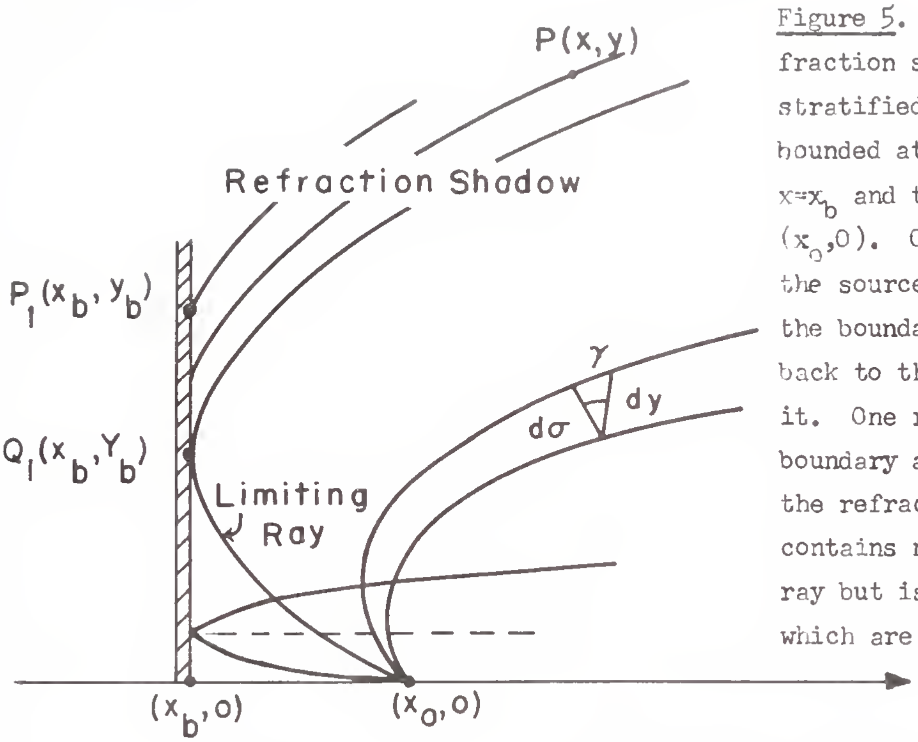

Figure 5

Figure 5. Diffracted rays in a refraction shadow occurring in a plane stratified medium. The medium is bounded at the left by the plane $x=x_b$ and the source is located at $(x_0,0)$. Of the incident rays from the source some are reflected from the boundary and others are refracted back to the right before reaching it. One ray is just tangent to the boundary and the region above it is the refraction shadow. This region contains no incident or reflected ray but is covered by diffracted rays which are congruent to the tangent ray. They originate from the tangent ray at its point of tangency, travel along the boundary and then leave it tangentially.

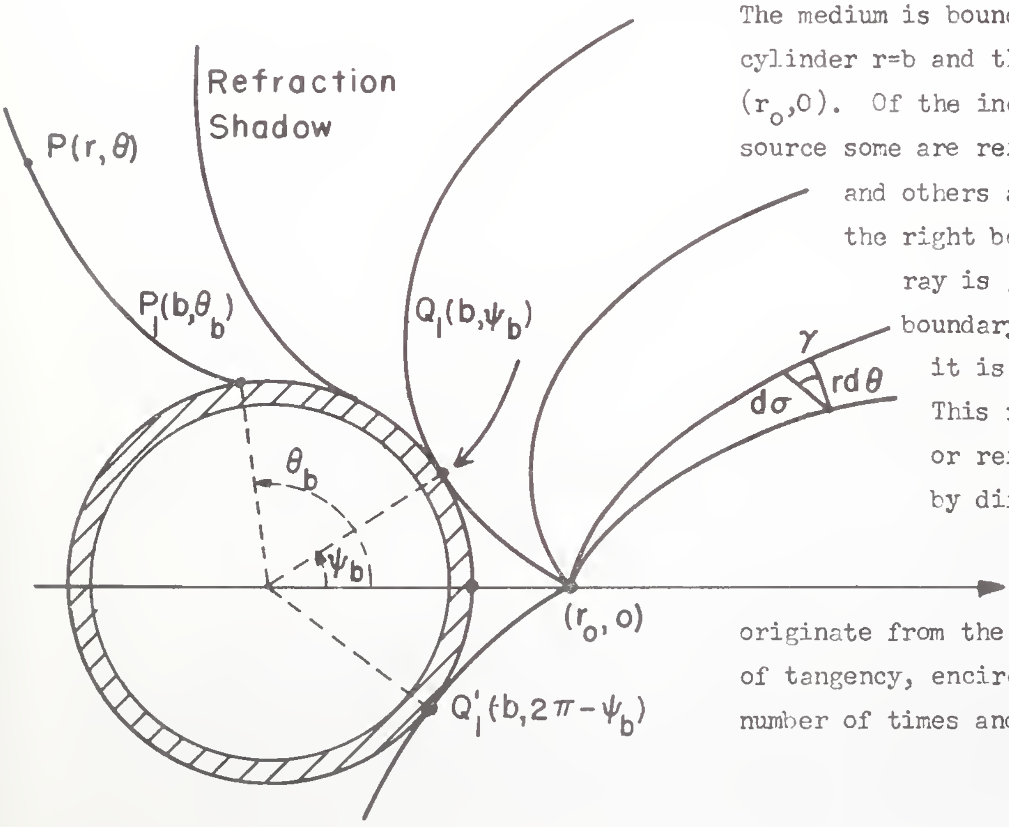

Figure 6

Figure 6. Diffracted rays in a refraction shadow occurring in a cylindrically stratified medium. The medium is bounded internally by the cylinder $r=b$ and the source is located at $(r_0,0)$. Of the incident rays from the source some are reflected from the boundary and others are refracted back to the right before reaching it. One ray is just tangent to the boundary and the region above it is the refraction shadow. This region contains no incident or reflected ray but is covered by diffracted rays which are congruent to the tangent ray. They originate from the tangent ray at its point of tangency, encircle the boundary any number of times and then leave it tangentially.

6. The field due to a point source in a plane stratified medium with a plane boundary

We now consider the field of a point source situated at $(x_o,0,0)$. The index of refraction of the plane stratified medium is again $n = n(x)$ and the medium is bounded on the left by the plane $x = x_b$ on which the field satisfies (49).

The rays are still plane curves lying in the planes $\ell y + z = 0$. They are determined by (23) plus the equation

$$(67) \qquad \frac{d^2 z}{dt^2} = 0, \qquad z = bt + z_o.$$

Now, the field must be rotationally symmetric about the x-axis. Therefore, for each value of $\ell = \frac{b}{a}$ = const., the rays in the plane $\ell y + z = 0$ are characterized by the second equation of (23) and

$$(68) \qquad r = \tilde{a}\, t.$$

Here, $\tilde{a} = a(1 + \ell^2)^{1/2}$ and $r^2 = y^2 + z^2$. Therefore, we can make use of all the results of the previous section if we merely replace a by $\tilde{a}$ and y by r. The only modification occurs in the amplitude calculation. The amplitude for any direct ray is given by (5) multiplied by $(r_o/r)^{1/2}$. $d\sigma_o/d\sigma$ is still interpreted as the limit of the two-dimensional area ratio. For the amplitude we thus obtain (51) multiplied by $r^{-1/2}$ and with a replaced by $\tilde{a}$. If we take the point source to be of unit strength*, then A_o has the value

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*By a point source of unit strength at $(x_o,0,0)$, we mean a source which produces locally the field $e^{ikn(x_o)R}/4\pi R$ where R is the distance from the source.

$$A_o = \tilde{a}^{1/2}(4\pi)^{-1}\left[n^2(x_o) - \tilde{a}^2\right]^{-1/4}.$$

Upon taking account of the foregoing remarks we finally obtain the following expressions for the incident and reflected fields:

$$(69) \quad u_i = \frac{a^{1/2}}{4\pi}\left[n^2-a^2\right]^{-1/4}\left[n^2(x_o) - a^2\right]^{-1/4}\left[r\int_{x_<}^{x_>}\frac{n^2dx}{(n^2-a^2)^{3/2}}\right]^{-1/2}$$

$$\cdot \exp\left[ik\int_{x_<}^{x_>}(n^2-a^2)^{1/2}dx + ikar\right],$$

$$(70) \quad u_r = \frac{R\,a^{1/2}}{4\pi}\left[n^2-a^2\right]^{-1/4}\left[n^2(x_o)-a^2\right]^{-1/4}\left[r\left\{\int_{x_b}^{x_o} + \int_{x_b}^{x}\right\}\frac{n^2dx}{(n^2-a^2)^{3/2}}\right]^{-1/2}$$

$$\cdot \exp\left[ik\left\{\int_{x_b}^{x_o} + \int_{x_b}^{x}\right\}(n^2-a^2)^{1/2}dx + ikar\right].$$

To find the field along the diffracted rays, we again utilize (14)-(21). This time the surface rays are radial lines in the plane $x = x_b$ with the point $(x_b,0,0)$ as center. Let $(x_b,r_b) = P_1$ and $(x_b,R_b) = Q_1$ be respectively the points of shedding and of grazing incidence. Then, on the plane $d\sigma_o/d\sigma = r_b/R_b$. If we use (5) modified as above, we find instead of (60) that the amplitude of the diffracted field is given by

$$(71) \quad A_d(P) = A_i(Q_1)D_m^2\left[n^2-n^2(x_b)\right]^{-1/4}\left[\frac{n(x_b)\;K\;(x_b)r_b}{r}\right]^{1/2} e^{-\alpha_m(r_b-R_b)}.$$

Now along the surface ray the phase change is

$$(72) \quad n(x_b)(r_b - R_b).$$

Combining (72) with the phase on the diffracted ray off the surface and with $u_i(Q_1)$ which can be determined from (69), we obtain for the diffracted field

$$u_d = \sum_m \frac{D_m^2}{4\pi r^{1/2}} \left[n^2-n^2(x_b)\right]^{-1/4} \left[n^2(x_o)-n^2(x_b)\right]^{-1/4} n(x_b)\ K\ (x_b) \tag{73}$$

$$\exp\left[ik \left\{\int_{x_b}^{x} + \int_{x_b}^{x_o}\right\} \left[n^2-n^2(x_b)\right]^{1/2} dx + ikn(x_b)r - \alpha_m(r_b-P_b)\right].$$

D_m and α_m are again given by (65) and (66) which are obtained from (20) and (21) by letting $b \to \infty$. The result (73) for the diffracted field agrees exactly with (62) in Part II, which is the asymptotic expansion of the exact solution of the corresponding boundary value problem.

7. The field of a line source in a cylindrically stratified medium with a circular boundary

Let us now consider the field due to a line source parallel to the z-axis and located at $(r_o,0)$. The index of refraction is $n = n(r)$ and tends monotonically to the constant n_1 as $r \to \infty$. The field satisfies the impedance condition (49) with $\partial u/\partial x$ replaced by $\partial u/\partial r$ on the circular boundary $r = b$.

In this medium (1) and (2) reduce to

$$\text{(a)}\quad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d}{dr}\left(\frac{1}{2}n^2\right),$$

$$\text{(74)}$$

$$\text{(b)}\quad r\,\frac{d^2\theta}{dt^2} + 2\,\frac{d\theta}{dt}\,\frac{dr}{dt} = 0$$

and

(75) $$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = n^2.$$

A first quadrature of (74b) can be performed and yields

(76) $$\frac{d\theta}{dt} = \frac{a}{r^2}.$$

This can be inserted into (74a) and the resulting equation can be integrated to yield

(77) $$\frac{dr}{dt} = \pm\left(n^2 - \frac{a^2}{r^2}\right)^{1/2}.$$

Thus from (76) and (77) we have

(78) $$\frac{dr}{d\theta} = \pm\frac{r}{a}\left(r^2n^2-a^2\right)^{1/2}.$$

Again, a may be identified with the ray slope tan α at the source and then $a = \pm r_o n(r_o)\sin\alpha$, (see Figure 6).

In the previous problems, the zeros of n^2-a^2 (turning points) played an essential role. The analog of such a zero is now any solution of

(79) $$r^2n^2(r) - a^2 = 0.$$

These zeros include the points of closest approach of a ray to the origin, as well as inflection points. Consequently for each of the two limiting rays which meet the circular boundary at grazing incidence, the points of tangency are zeros of (79). If $\pm\alpha_b$ corresponds to the limiting rays (79) yields

(80) $$a = bn(b) = \pm r_o n(r_o)\sin\alpha_b.$$

From (78) we find for the equations of the incident rays

$$(81) \qquad \theta = \int_{r_<}^{r_>} \frac{a\,dr}{r(r^2n^2-a^2)^{1/2}} \, .$$

Here $r_>$ and $r_<$ are respectively the larger and smaller of r and r_o. To find the phase along these rays, we apply (3) in conjunction with (76) - (78) and ascertain that

$$(82) \qquad \phi = a\theta + \int_{r_<}^{r_>} (n^2 - \frac{a^2}{r^2})^{1/2} \, dr.$$

The amplitude is now found by using (5). First we note that along an incident ray, we have

$$\frac{d\sigma}{d\theta} = r \frac{dr}{d\theta} \left[r^2 + (\frac{dr}{d\theta})^2 \right]^{-1/2} .$$

By making use of this expression, we find that

$$(83) \qquad \frac{d\sigma_o}{d\sigma} = \frac{n\left[r_o^2n^2(x_o) - a^2\right]^{1/2}(\partial\theta_o/\partial a)}{n(x_o)\left[r^2n^2-a^2\right]^{1/2}(\partial\theta/\partial a)} \, .$$

The derivatives in (83) can be found from (81). If we insert (83) into (5) and lump all constant terms together, we obtain for the amplitude

$$(84) \qquad A = A_o\left[r^2n^2-a^2\right]^{-1/4} \left[\int_{r_<}^{r_>} \frac{rn^2dr}{(r^2n^2-a^2)^{3/2}} \right]^{-1/2} .$$

The constant A_o in (84) is proportional to the field strength at the source. If the source is of unit strength, then

$$(85) \qquad A_o = (\frac{i}{8\pi k})^{1/2} \left[r_o^2n^2(r_o) - a^2 \right]^{-1/4} .$$

Upon collecting the above results, we obtain for the incident field

$$(86)\qquad u_i = \left[r_o^2 n^2(r_o)-a^2\right]^{-1/4}\left[r^2n^2-a^2\right]^{-1/4}\left[8\pi k\int_{r_<}^{r_>}\frac{rn^2dr}{(r^2n^2-a^2)^{3/2}}\right]^{-1/2}$$

$$\cdot\exp\left[ik\int_{r_<}^{r_>}\left[n^2-\frac{a^2}{r^2}\right]^{1/2}dr + ika\theta + \frac{i\pi}{4}\right].$$

Those rays which meet the boundary non-tangentially give rise to a family of reflected rays with an associated reflected field. The details of the construction of this field will not be given since they are similar to those employed in the two problems of Sections 5 and 6.

Let us now consider a ray for which (79) has a solution $r = r_a$. Such a ray has r_a as its distance of closest approach to the origin and after reaching this point it turns away from the origin. The equations of such a ray and the phase and amplitude on it up to $r = r_a$ are given by (81), (82) and (84) respectively. However, (84) becomes indeterminate at $r = r_a$. In the limit, as r approaches r_a, (84) yields

$$(89)\qquad A = A_o\left[1 + r_a\frac{n'(r_a)}{n(r_a)}\right]^{1/2} = A_o\left[1 + r_a K(r_a)\right]^{1/2}.$$

$K(r_a)$ is again the curvature of the ray at $r = r_a$. On the outgoing portion of the ray for which $r > r_a$ the ray is given by

$$(90)\qquad \theta = \left\{\int_{r_a}^{r_o} + \int_{r_a}^{r}\right\}\frac{a\,dr}{r(r^2n^2-a^2)^{1/2}}.$$

The associated phase is

$$(91) \qquad \phi = a\theta + \left\{ \int_{r_a}^{r_o} + \int_{r_a}^{r} \right\} \left(n^2 - \frac{a^2}{r^2}\right)^{1/2} dr.$$

The amplitude is given by an expression of the form (84) with the integral replaced by the sum of two integrals. The limits are from r_a to r and r_a to r_o respectively.

We now turn to the evaluation of the diffracted field in the refraction shadow formed by the circular boundary and the continuation of the limiting ray. The initial direction of the diffracted rays is that of the surface ray on the boundary. This ray is just the boundary itself. All the diffracted rays which emanate from it are congruent. Any two can be made to coincide by a rotation of one of them. When we use (80) in (81) we find for the equations of the diffracted rays

$$(92) \qquad \theta = \int_{b}^{r} \frac{bn(b)dr}{r\left[r^2n^2 - b^2n^2(b)\right]^{1/2}} + \theta_b .$$

Here $(b,\theta_b) = P_1$ is the point of shedding of the ray through $(r,\theta) = P$. Through each point P in the shadow there will pass a double infinity of diffracted rays. The surface ray which arises at the point $(b,\Psi_b) = Q_1$ will encircle the cylinder infinitely many times in the counterclockwise sense. After each encirclement, it will shed a diffracted ray at P_1 and this ray will pass through P. Another surface ray will arise at $(b,-\Psi_b) = Q_1'$ going clockwise. It will also shed diffracted rays at $(b,\theta_b') = P_1'$ which will pass through P.

The field along these rays is obtained by using (14)-(19). We first determine the field due to the rays which travel counterclockwise. The phase along one of the diffracted rays which arises after the surface ray has made q counterclockwise circuits of the boundary is given by

$$(93) \qquad \phi = bn(b)\theta + 2\pi qbn(b) + \left\{ \int_b^r + \int_b^{r_o} \right\} \left[n^2 - \frac{b^2n^2(b)}{r^2} \right]^{1/2} dr.$$

Because the surface rays are parallel curves, $d\sigma_o/d\sigma = 1$. Upon carrying out the limit in (15), we find for the amplitude on a diffracted ray the result

$$(94) \qquad A_d(P) = A_d'(P_1)\left[r^2n^2-b^2n^2(b)\right]^{-1/4} \left[bn(b) \left\{K(b)+b^{-1}\right\} \right]^{1/2} .$$

To determine $A_d'(P_1)$ we note that the index of refraction and the radius of curvature are constant along the boundary so that D_m and α_m are constant. Upon evaluating $A_d'(P_1)$ and inserting it into (94) we find for the amplitude on a ray which leaves the surface after q circuits the result

$$(95) \qquad A_d(P) = A_i(Q_1)D_m^2\left[bn(b) \left\{K(b) + b^{-1}\right\} \right]^{1/2} \left[r^2n^2-b^2n^2(b)\right]^{-1/4} \exp\left[-\alpha_m(\tau+2\pi qb)\right] .$$

The value of $A_i(Q_1)$ can be obtained from (85) and (89).

When we collect the above results we find for the diffracted field corresponding to each mode a series of terms, one for each value of q. This is a geometric series which can be summed explicitly. Upon summing the series we obtain for the diffracted field u_d^+ produced by one of the tangent incident rays the result

$$(96) \qquad u_d^+ = \left[r^2n^2-b^2n^2(b)\right]^{-1/4}\left[r_o^2n^2(r_o)-b^2n_b^2\right] \frac{n^{1/2}(b)b\left[b^{-1}+K(b)\right]}{(8\pi k)^{1/2}}$$

$$\cdot \exp\left[ik\left\{ \int_b^r + \int_b^{r_o} \right\} \left[n^2 - \frac{b^2n^2(b)}{r^2}\right]^{1/2} dr + \frac{\pi i}{4}\right]$$

$$\sum_m D_m^2 \left[1-\exp\left\{2\pi i\left[bkn(b)+ib\alpha_m\right]\right\}\exp\left\{ikbn(b)\theta-\alpha_m b(\theta_b-\Psi_b)\right\}\right] .$$

In (96) the diffraction coefficients D_m and the decay constants α_m are given by (20) and (21). In an analogous way we find the diffracted field u_d^- due to the other tangent incident ray which gives rise to the surface rays encircling the cylinder clockwise. We obtain an expression of the form (96) with the last exponential in the sum replaced by

$$(97) \qquad \exp\left[ikbn(b)(2\pi-\theta) - \alpha_m b(2\pi - 2\theta + \theta_b - \Psi_b)\right].$$

The total diffracted field u_d is given by $u_d = u_d^+ + u_d^-$.

The above result for the diffracted field agrees exactly with (77) of Part II obtained by asymptotic expansion of the exact solution of the corresponding boundary value problem.

8. Trapped modes in a plane duct

We now consider a medium in which the index of refraction varies as in Figure 1b. If a source placed at infinity between x_a and x_a' produces rays with a given value of a, these rays remain trapped between the lines $x=x_a$ and $x=x_a'$. The points x_a, x_a' are turning points for the rays and these lines are then caustics for them. Two such rays will pass through any point for which $x_a < x < x_a'$. This corresponds to the fact that when the parameter $a < n_{max}$, the quantity $(n^2-a^2)^{1/2}$ is pure imaginary for $x < x_\alpha$ or $x > x_\alpha'$. Thus, (24) will have real solutions only in the interval $x_\alpha \leq x \leq x_\alpha'$.

Let x_o be the zero of $n'(x)$ which lies between the two turning points. The two rays passing through $(x_o,0)$ and proceeding towards $y = \infty$, are characterized by the equations

$$y = \left\{ \int_{x_o}^{x_a'} + 2[q/2] \int_{x_a}^{x_a'} + (-1)^{q+1} \int_{x}^{x_a'} \right\} \frac{a dx}{(n^2-a^2)^{1/2}}$$

(98)

$$y = \left\{ \int_{x_a}^{x_o} + 2[q/2] \int_{x_a}^{x_a'} + (-1)^{q+1} \int_{x_a}^{x} \right\} \frac{a\ dx}{(n^2-a^2)^{1/2}} .$$

Here $[q/2]$ is the greatest integer less than or equal to $q/2$, and q is an integer related to the number of times the ray has gone through the caustic and turned back after leaving $(x_o,0)$ and before passing through $P = (x,y)$.

The field at P is the sum of the fields on the two rays through P

(99) $$u = A_1(P)e^{ik\phi_1(P)} + A_2(P)e^{ik\phi_2(P)} .$$

In the strip $x_a < x < x_a'$, $\nabla\phi$ is double valued and ϕ has infinitely many values. However, only two of the branches of ϕ are essentially distinct, the others differing from these by additive constants. The amplitude A(P) also has the same behavior. Now the field given in (99) must be single-valued at P and we assume that this is true of each term separately. Then the difference between corresponding branches of the functions ϕ and A, $\delta\phi$ and $\delta \log A$ respectively, must satisfy

(100) $$k\delta\phi - i\ \delta \log A = 2\pi m .$$

Here m is an integer.

A covering surface can be associated with $\nabla\phi$ consisting of two sheets joined along the lines defining the caustics, since along these lines $\nabla\phi$ is single-valued. The determinations $\nabla\phi_1$ and $\nabla\phi_2$ are associated with points on the upper and lower sheets respectively. On the same surface $\nabla \log A$ is also single valued.

If $\vec{t}$ is the unit tangent along a curve on the covering surface, and if arclength $s = 0$ corresponds to a point P_o, then at any point P ϕ is given by

$$(101) \qquad \phi(P) = \phi(P_o) + \int_0^{s(P)} \nabla\phi \cdot \vec{t}\, ds + m\oint \nabla\phi \cdot \vec{t}\, ds.$$

The last integral is taken around any closed path starting at P. Therefore the changes in ϕ and log A can be written

$$(102) \qquad \delta\phi = \oint \nabla\phi \cdot \vec{t}\, ds$$

$$(103) \qquad \delta \log A = \oint \nabla \log A \cdot \vec{t}\, ds.$$

The change in log A corresponds to the number of times a ray goes through a caustic in the direction of increasing phase. In making a complete circuit along the covering surface, a ray goes through a caustic twice so that $\delta(\log A) = -\pi$. Inserting this and (102) into (100), we obtain

$$(104) \qquad k \oint \nabla\phi \cdot \vec{t}\, ds - \pi = 2\pi m.$$

To evaluate the integral, we take the closed path determined by the ray which goes along the upper sheet from P_o on $x = x_\alpha'$ to P_1 on $x = x_\alpha$ and then along the lower sheet to P_2 and finally back along the caustic to P_o. Now along the caustic

$$(105) \qquad \int_{P_2}^{P_o} \nabla\phi \cdot \vec{t}\, ds = a\left[y(P_o) - y(P_2)\right].$$

If we use (98) to simplify this and to evaluate the remaining integral, we find

$$(106) \qquad \oint \nabla\phi\cdot\vec{t}\, ds = -\int_{x_a'}^{x_a} \frac{n^2\, dx}{(n^2-a^2)^{1/2}} + \int_{x_a}^{x_a'} \frac{n^2 dx}{(n^2-a^2)^{1/2}} - a\left[y(P_2) - y(P_o)\right]$$

$$= 2\int_{x_a}^{x_a'} (n^2-a^2)^{1/2}\, dx.$$

Upon inserting (106) into (104) we obtain the equation

$$(107) \qquad k \int_{x_a}^{x_a'} (n^2-a^2)^{1/2}\, dx = (m + \tfrac{1}{2})\, \pi \, .$$

This equation determines the possible values of a for each given value of m. According to (99) and (101) the phase of each term in u is kay + ... Thus ka is the propagation constant in the y direction.

In part II the problem of trapped modes is formulated as an eigenvalue problem, the eigenvalue being the proportion constant in the y direction divided by k. The asymptotic solution of this eigenvalue is found to be given by (19). This equation coincides exactly with the above equation for a. This agreement constitutes a partial justification of the geometrical method introduced here.

Part II. Asymptotic Expansion of Solutions of Boundary Value Problems

1. Introduction

In Part I, a geometrical method was given for calculating fields. The method was also used to determine the fields explicitly in various diffraction problems. In this section, we consider these same diffraction problems formulating them as boundary value problems for the equation

$$(1) \qquad \left[\Delta + k^2 n^2(x)\right] u = 0.$$

We then solve these problems explicitly and expand the solutions asymptotically with respect to k as $k \to \infty$. The leading term of the asymptotic expansion which we call the asymptotic form of the solution is all we consider, in general. We find in every case that the asymptotic form of the solution of the boundary value problem agrees exactly with the field determined by the geometric theory. This agreement constitutes a partial demonstration of the validity of the geometric method of Part I.

In calculating the exact solution asymptotically, the WKB method and the method of stationary phase are applied. Where the solution is given by an integral, the method of stationary phase yields the asymptotic form of the field whenever there exists a stationary point. This method gives the field in the lit region. For those field points which lie in the refraction shadow, there are no stationary points. We then transform the integral into a Fourier-Bessel series by making a Watson transformation. The subsequent analysis then yields the field as a sum of modes.

In the analysis of diffraction in inhomogeneous media, similar techniques have been employed by C. L. Pekeris[5] for an acoustical point source, by F. G. Friedlander[6] for a pulse and by N. A. Haskell[7]. D.C. Pridmore Brown and A. Ingard[8] have verified that the asymptotic form agrees with the measured diffraction field in the refraction shadow for sound in a stratified medium with a constant temperature gradient.

2. Plane waves and trapped modes in a plane stratified medium

Let us first consider a plane stratified medium for which $n = n(x)$. We will assume that $n(x)$ tends to the constant n_o as x tends to $-\infty$ and to n_1 as x tends to $+\infty$. Now suppose a wave $\exp[ikn_o(x \cos\alpha + y \sin\alpha)]$ is incident from $-\infty$. Then the field u will be proportional to $\exp[ikn_o\, y \sin\alpha]$ and independent of z. Therefore we write u in the form

$$u(x,y) = e^{ikn_o y \sin\alpha} f(x) . \tag{2}$$

We suppose that at $-\infty$, $f(x)$ behaves like

$$f(x) \sim e^{ikn_o x\cos\alpha} + R\, e^{-ikn_o x\cos\alpha} \tag{3}$$

for some value of the constant R. At $x = +\infty$ we require $f(x)$ to satisfy the radiation condition,

$$\lim_{x\to\infty} |f_x - ikn_1 \cos\alpha\, f| = 0 . \tag{4}$$

Upon inserting (2) into (1) we find that f satisfies

$$f_{xx} + k^2(n^2 - n_o^2 \sin^2\alpha)f = 0 . \tag{5}$$

We now seek that solution of (5) which satisfies (3) and (4). We will obtain the asymptotic form for large k of that solution. Then we will insert it into (2) to obtain the asymptotic form of $u(x,y)$.

A point at which the coefficient of f in (5) vanishes is called a turning point. If there is no turning point and $n^2 > n_o^2 \sin^2\alpha$, then the asymptotic form of f can be found by the WKB method. Upon inserting it into (2) we obtain

$$u(x,y) \sim \frac{(n_o \cos\alpha)^{1/2}}{(n^2 - n_o^2 \sin^2\alpha)^{1/4}} \exp\left[ikn_o(x \cos\alpha + y \sin\alpha) + ik\int_{-\infty}^{x} g(x)dx\right] . \tag{6}$$

Here $g(x)$ is defined by

$$(7)\qquad g(x) = (n^2 - n_o^2 \sin^2\alpha)^{1/2} - n_o \cos\alpha \ .$$

If there is just one turning point at $x = x_\alpha$ and if $n^2 > n_o^2 \sin^2\alpha$ for $x < x_\alpha$ then for $x \neq x_\alpha$ the asymptotic form of any solution of (5) is

$$(8)\quad f(x) \sim \begin{cases} (n^2-n_o^2 \sin^2\alpha)^{-1/4} \left[A \cos(k\zeta(x)-\frac{\pi}{4}) + B \cos(k\zeta(x)+\frac{\pi}{4})\right] & x < x_\alpha \\ (n_o^2 \sin^2\alpha-n^2)^{-1/4} \left[\frac{A}{2} e^{-k\eta(x)} + B e^{k\eta(x)}\right] & x > x_\alpha \ . \end{cases}$$

Here A and B are constants while ζ and η are defined by

$$(9)\qquad \zeta(x) = \int_x^{x_\alpha} (n^2-n_o^2 \sin^2\alpha)^{1/2}\, dx$$

$$(10)\qquad \eta(x) = \int_{x_\alpha}^{x} (n_o^2 \sin^2\alpha-n^2)^{1/2}\, dx \ .$$

Upon imposing the radiation condition (4) we see that $B = 0$. Equation (3) determines the value of A. When these constants are inserted into (8) and the result is substituted into (2), the following expansion of u for $x \neq x_\alpha$ results

$$(11)\quad u \sim \frac{(n_o \cos\alpha)^{1/2}}{(n^2-n_o^2 \sin^2\alpha)^{1/4}} \left\{ \exp\left[ikn_o(x\cos\alpha + y\sin\alpha) + ik\int_{-\infty}^{x} g(x)dx\right] \right.$$

$$\left. + \exp\left[ikn_o(-x\cos\alpha+y\sin\alpha)-ik\int_{-\infty}^{x} g(x)dx+2ik\int_{-\infty}^{x_\alpha} g(x)dx+2ikn_o x_\alpha \cos\alpha- i\pi/4\right]\right\},\quad x < x_\alpha \ .$$

$$(12)\quad u\sim\frac{(n_o\cos\alpha)^{1/2}}{(n_o^2\sin^2\alpha-n^2)^{1/4}}\exp\left[ikn_o\, y\sin\alpha - k\eta(x)+ik\int_{-\infty}^{x_\alpha} g(x)dx+ikn_o x_\alpha\cos\alpha-i\pi/4\right],\quad x > x_\alpha .$$

Now let us consider the case in which there are exactly two turning points, one at x_α and the other at $x'_\alpha > x_\alpha$. Then the second asymptotic form of f in (8) applies only for $x < x'_\alpha$. Two other forms, corresponding to those in (8), apply in the regions $x'_\alpha < x$ and $x_\alpha < x < x'_\alpha$ respectively. By matching the constants of the two forms which both apply in the interval between the turning points, and applying (3) and (4), we completely determine the asymptotic form of f. When this is inserted into (2), the asymptotic form of u is obtained. In the region $x < x_\alpha$ this form can be obtained from (11) by multiplying the second exponential by the factor $e^{-i\pi/4}\left[4-e^{-2k\eta(x'_\alpha)}\right]\left[4+e^{-2k\eta(x'_\alpha)}\right]^{-1}$. In the interval between the turning points we have

$$(13)\qquad u\sim\frac{2(n_o\cos\alpha)^{1/2}\, e^{k\eta(x'_\alpha)}}{(n_o^2\sin^2\alpha-n^2)^{1/4}\left(1+4e^{2k\eta(x'_\alpha)}\right)}\exp\left[ikn_o(y\sin\alpha+x_\alpha\cos\alpha)+ik\int_{-\infty}^{x_\alpha} g(x)dx\right]$$

$$\cdot\left\{\exp\left[k\eta(x)+i\pi/4\right]+2\exp\left[-k\eta(x)-i\pi/4\right]\right\} .$$

For $x > x'_\alpha$ we have

$$(14)\qquad u\sim\frac{4(n_o\cos\alpha)^{1/2}\, e^{k\eta(x'_\alpha)}}{(n^2-n_o^2\sin^2\alpha)^{1/4}\left(1+4e^{2k\eta(x'_\alpha)}\right)}\exp\left[ikn_o(y\sin\alpha+x_\alpha\cos_\alpha)+ik\int_{-\infty}^{x} g(x)dx\right.$$

$$\left.+ik\int_{x'_\alpha}^{x}(n^2-n_o^2\sin^2\alpha)^{1/2}dx\right] .$$

These results have been compared with those of the geometrical theory in Part I, Section 4.

Now let us consider the trapped modes in a plane stratified medium. These modes are solutions of the form

$$u(x,y) = e^{ikay} f(x) . \tag{15}$$

In (15) a is a constant which is to be determined. The product ka is called the propagation constant of the mode in the y direction. The function f(x) is seen to satisfy

$$f_{xx} + k^2(n^2-a^2) f = 0. \tag{16}$$

We also require f(x) to satisfy the radiation conditions

$$\lim_{x \to \infty} |f_x - ik (n_1^2-a^2)^{1/2} f| = 0 \tag{17}$$

$$\lim_{x \to -\infty} |f_x + ik (n_0^2-a^2)^{1/2} f| = 0. \tag{18}$$

It is clear that this problem will have non-zero solutions only for certain values of a, which are called eigenvalues.

We assume that $n^2(x)$ has the behavior shown in Figure 1b, with $n_0 < n_1$. It is readily seen from the asymptotic forms of f(x) constructed above that asymptotically there is no solution of (16) - (18) if $a^2 < n_1^2$ or $a^2 \geq n_{max}^2$. Therefore we assume that $n_1^2 < a^2 \leq n_{max}^2$. Then there are exactly two turning points in (16). We now construct the asymptotic form of f(x) in the same way as before. The two asymptotic forms which both apply between the two turning points do not match unless a^2 satisfies the equation

$$k\int_{x_a}^{x_a'} (n^2-a^2)^{1/2} \, dx = \left(m + \frac{1}{2}\right) \pi . \tag{19}$$

Here x_a and x'_a denote the turning points, which are the two roots of $n^2-a^2=0$, and m denotes any non-negative integer. Equation (19) is the eigenvalue equation and its roots a_m are the eigenvalues.

The eigenvalue equation (19) can be solved asymptotically for fixed m and k large. For as k becomes infinite the integral must tend to zero. Since the integrand is positive, except at the endpoints, the length of the interval of integration must tend to zero. Therefore x_a and x'_a must both tend to x_{max}, the value of x at which n assumes its maximum. Upon making use of this fact we obtain the results

$$(20) \qquad a_m \sim n_{max} - \left(\frac{-n''_{max}}{n_{max}}\right)^{1/2} \frac{m+1/2}{k} \quad .$$

The value of x_a is found to be

$$(21) \qquad x_a \sim x_{max} - \left(\frac{2m+1}{k}\right)^{1/2} (-n_{max}\, n''_{max})^{-1/4} \quad .$$

The value of x'_a is obtained from (21) by changing the sign of the second term. The asymptotic forms of f(x) and of u(x,y) are now easily obtained but will not be given here. The above results for a_m, x_a and x'_a have been compared with those of the geometrical theory in Part I, Section 8.

3. The field of a line source in a plane stratified medium with a plane boundary

Let us again consider a plane stratified medium bounded by a plane at $x=x_b$. Suppose a line source parallel to the z-axis is located at $(x_0,0)$. Then the field u will be independent of z. As before, the index n(x) tends to n_1 as x tends to $+\infty$. The field again satisfies (1) with the inhomogeneous term $-\delta(x-x_0)\,\delta(y)$ on the right side for $x > x_b$. On the boundary we require that u satisfy the impedance condition

$$\frac{\partial u}{\partial x} = ikZu \quad . \tag{22}$$

Here the constant Z is the impedance of the boundary plane. If ρ denotes distance from the source in the x,y plane then as $\rho \rightarrow \infty$, u must satisfy the radiation condition

$$\lim_{\rho \rightarrow \infty} \rho^{1/2} \left| u - iku_\rho \right| = 0 \quad . \tag{23}$$

To solve (1) (with the source term), (22) and (23) for u we introduce $f(x,a)$, the Fourier transform of $u(x,y)$ with respect to y, defined by

$$f(x,a) = \int_{-\infty}^{\infty} u(x,y)\, e^{-ikay}\, dy \quad . \tag{24}$$

From the formulation of the problem it follows that $u(x,y) = u(x, -y)$. Then (24) shows that $f(x,a) = f(x, -a)$. Upon taking the Fourier transform of (1) (with the source term) we find that $f(x,a)$ satisfies (16) with the inhomogeneous term $-\delta(x-x_o)$ on the right side. Let f_1 and f_2 be two linearly independent solutions of the homogeneous equation (16) which are even functions of a. We require f_1 to satisfy the radiation condition (17) and f_2 to satisfy the boundary condition (22). Then we attempt to construct f in the form

$$f(x,a) = \begin{cases} c_1\, f_1\, (x,a) & x_o \leq x < \infty \\ c_2\, f_2\, (x,a) & x_b \leq x \leq x_o \quad . \end{cases} \tag{25}$$

This form for f satisfies (16) for $x \neq x_o$, and satisfies (17) and (22).

By taking account of the term $-\delta(x-x_o)$ in the differential equation for f, and requiring f to be continuous at $x=x_0$ we can determine the constants c_1 and c_2 in (25). Then (25) becomes

$$f(x,a) = f_1(x_>,a)\, f_2(x_<,a)\, W^{-1}(a) \quad . \tag{26}$$

Here $x_>$ and $x_<$ indicate respectively the larger and smaller of the values x and x_0. The function W(a) is the Wronskian of $f_1(x,a)$ and $f_2(x,a)$ which is independent of x but which depends upon a. According to (26) and the evenness of f_1 and f_2, f is even in a as it must be. Now we take the inverse transform of f(x,a) and obtain

$$u(x,y) = \frac{k}{2\pi}\int_{-\infty}^{\infty} f(x,a)\, e^{ikay}\, da \ . \tag{27}$$

It may be verified by direct computation that (27) is the solution of the problem. For our subsequent calculations it is convenient to write $u = I^+ + I^-$ where

$$I^{\pm} = \frac{k}{2\pi}\int_{0}^{\infty} f(x,a)\, e^{\pm ikay}\, da. \tag{28}$$

We will now evaluate u(x,y) asymptotically for large k. Two different methods of evaluation will be necessary, one for points x,y in the illuminated region and the other for points in the shadow region. For points in the illuminated region the integrals in (28) will be evaluated by the method of stationary phase. The stationary points will be found to correspond to the rays through the point. One will correspond to the direct ray from the source to x,y and another to the reflected ray. For points in the shadow region there will be no stationary points. Therefore the integrals will be evaluated differently, namely, in terms of residues at the poles of the integrands in the complex a-plane.

To perform the stationary phase calculation, we must now evaluate f(x,a) asymptotically for large k. We do this by first evaluating f_1 and f_2 asymptotically as we did in the previous section. For this purpose it is convenient to subdivide the a-axis into the three intervals 0 to $n(x_b)$, $n(x_b)$ to n_1 and n_1 to ∞. In the first and third intervals there are no turning points for any value of x, while in the second interval there is exactly one turning point x_a for each x. Therefore, for a in the first interval

$$(29) \qquad f_1(x,a) \sim c(n^2-a^2)^{-1/4} \exp\left[ik \int_{x_b}^{x} (n^2-a^2)^{1/2} dx\right]$$

$$(30) \qquad f_2(x,a) \sim c'(n^2-a^2)^{-1/4} \sin\left[k \int_{x_b}^{x} (n^2-a^2)^{1/2} dx + \gamma\right].$$

Here c and c' are arbitrary constants and γ is a solution of the equation

$$(31) \qquad \cot \gamma = iZ \left[n^2(x_b)-a^2\right]^{-1/2} .$$

For a in the interval $n(x_b) < a < n_1$, there are two cases. If the turning point $x_a < x_o$, f_1 is given by (29). When $x_b \leq x < x_a$, (30) and (31) still apply for f_2 provided n^2-a^2 is replaced by a^2-n^2, sin replaced by sinh in (30) and cot replaced by coth in (31). For $x_a < x \leq x_o$, we use this and (8) with $n_o \sin \alpha$ replaced by a to show that

$$(32) \quad f_2 \sim c'(n^2-a^2)^{-1/4} \left\{ e^{k\int_{x_b}^{x_a} (a^2-n^2)^{1/2} dx + \gamma} \sin\left(k \int_{x_a}^{x} (n^2-a^2)^{1/2} dx + \frac{\pi}{4}\right) - \frac{1}{2} e^{-k\int_{x_b}^{x_a} (a^2-n^2)^{1/2} dx - \gamma} \cos\left(k \int_{x_a}^{x} (n^2-a^2)^{1/2} dx + \frac{\pi}{4}\right) \right\}.$$

On the other hand, if $x_a > x_o > x$, f_2 is again given by (30) and (31) with the above changes. f_1 is given by (29) for $x > x_a$ provided x_b is replaced by x_a. For $x_o \leq x < x_a$, we again use (8) and show that

$$(33)\qquad f_2 \sim c(a^2-n^2)^{-1/4}\left\{\frac{1}{2}\, e^{-k\int_x^{x_a}(a^2-n^2)^{1/2}dx + \frac{\pi i}{4}} + e^{k\int_x^{x_a}(a^2-n^2)^{1/2}dx - \frac{\pi i}{4}}\right\}.$$

Let the integrals corresponding to each of the a-intervals be denoted by I_1^+, I_2^+, I_3^+ respectively. The integrand of I_1^+ may be expanded asymptotically by using (29) and (30) for f_1 and f_2 in (26). This yields

$$(34)\qquad I_1^+ \sim \frac{i}{4\pi}\int_0^{n(x_b)} \frac{\left\{\exp\left(ik\int_{x_<}^{x_>}(n^2-a^2)^{1/2}dx+ikay\right) + R\exp\left(ik\left\{\int_{x_b}^{x} + \int_{x_b}^{x_o}\right\}(n^2-a^2)^{1/2}dx+ikay\right)\right\}da}{\left[n^2(x_o)-a^2\right]^{1/4}\qquad (n^2-a^2)^{1/4}}.$$

Here, the quantity R is defined by

$$(35)\qquad R = \frac{\left[n^2(x_b)-a^2\right]^{1/2} + Z}{\left[n^2(x_b)-a^2\right]^{1/2} - Z}.$$

We now apply the method of stationary phase to evaluate each of the two integrals of which I_1^+ is the sum. The stationary points a are given respectively by

$$(36)\qquad y - \int_{x_<}^{x_>} \frac{a\,dx}{(n^2-a^2)^{1/2}} = 0,$$

$$(37)\qquad y - \left\{\int_{x_b}^{x} + \int_{x_b}^{x_o}\right\} \frac{a\,dx}{(n^2-a^2)^{1/2}} = 0.$$

If the point (x,y) is in the lit region and $y \geq 0$, these two conditions can be satisfied for certain a. This is apparent because (36) and (37) are the equations of the incident and reflected rays discussed in Part I, Section 5. If these values of a lie in the first interval then, as we will see, there are no stationary points in I_2^+, I_3^+ or I^-. Therefore I^+ is given by the sum of the two contributions from the two stationary points of I_1^+. Denoting these contributions by u_i and u_r we have $u \sim u_i + u_r$ where

$$(38) \qquad u_i \sim \frac{\exp\left[ik \int_{x_<}^{x_>} (n^2-a^2)^{1/2}\, dx + ikay + \frac{\pi i}{4}\right]}{(n^2(x_0)-a^2)^{1/4}\, (n^2-a^2)^{1/4} \left(8\pi k \int_{x_<}^{x_>} \frac{n^2 dx}{(n^2-a^2)^{3/2}}\right)^{1/2}}$$

$$(39) \qquad u_r \sim \frac{R \; \exp\left[ik \left\{ \int_{x_b}^{x_0} + \int_{x_b}^{x} \right\} (n^2-a^2)^{1/2} dx + ikay + \frac{\pi i}{4}\right]}{(n^2(x_0)-a^2)^{1/4} (n^2-a^2)^{1/4} \left(8\pi k \left\{ \int_{x_b}^{x} + \int_{x_b}^{x_0} \right\} \frac{n^2 dx}{(n^2-a^2)^{3/2}}\right)^{1/2}} .$$

The asymptotic form of I_2^+, where the turning point $x_a < x_0$ and $x_a < x$, is given by (34). However R must be replaced by $e^{-\pi i/2}$ and x_b by x_a in the lower limit of the integrals in the integrand of I_1^+. With these changes, the stationary points are again given by (36) and (37). If the roots of the equations lie in the second interval then I_2^+ has two stationary points while I_1^+ and I_3^+ have none. In this case the asymptotic form of u is again given by (38) and (39).

When $x_a > x_0$ the asymptotic form of I_2^+ is obtained by using (29) and (33) for f_1 and the modified form of (30) for f_2. However, on applying the method of stationary phase to the resulting integral, we find that there are no stationary points. Either (36) and (37) correspond to parameter values a for which there are no rays or the integrand contains only real exponentials. Finally, the asymptotic form of I_3^+ for any (x,y) also involves real exponentials so that we obtain no field contributions from this integral.

When $y < 0$ the same results (38) and (39) are obtained from stationary points of I^-, while I^+ has none. The result $u \sim u_i + u_r$ has been compared with the geometrical result in Part I, Section 5.

Let us now consider a point (x,y) in the shadow region. For such a point none of the integrals considered above has a stationary point so the method of stationary phase does not apply. In order to obtain a representation of u which will turn out to be useful in the shadow region, we will rewrite $I^{\pm}$. From Figure 7 we see that

$$(40) \qquad \int_{OP} f e^{ikay}\, da = \int_{\Gamma_\pm + OQ_\pm + B_\pm} f e^{ikay}\, da \pm 2\pi i \sum_m R_m^{\pm} \quad .$$

A similar formula holds for the integral of fe^{-ikay}. Here $B_\pm$ denotes the possible branch cuts of $f(x,a)$ in the half-plane $\pm \mathrm{Im}\, a > 0$ and $R_m^{\pm}$ denotes the residue at the mth pole of the integrand in the same half-plane. In the Appendix it is shown that for $\pm y > 0$ the integral of $fe^{\pm ikay}$ over Γ_+ tends to zero as the radius of the semi-circle becomes infinite while for $\pm y < 0$ that over Γ_- tends to zero. In the Appendix it is shown that the integral of fe^{-ikay} over OQ_- is the negative of that of fe^{ikay} over OQ_+ asymptotically for large k. It is also shown that the branch cut integrals are asymptotically zero.

Upon making use of the above results in (27) and (28), and then using (26) we find that u is asymptotically given by the residue series

$$(41) \qquad u(x,y) \sim -ik \sum_m \frac{f_1(x_>, a_m) f_2(x_<, a_m) e^{ika_m|y|}}{\frac{\partial W(a_m)}{\partial a}} \quad .$$

In (41) the a_m are the roots of the equation

$$(42) \qquad W(a) = 0.$$

The residue series will now be approximated by using asymptotic expansions for

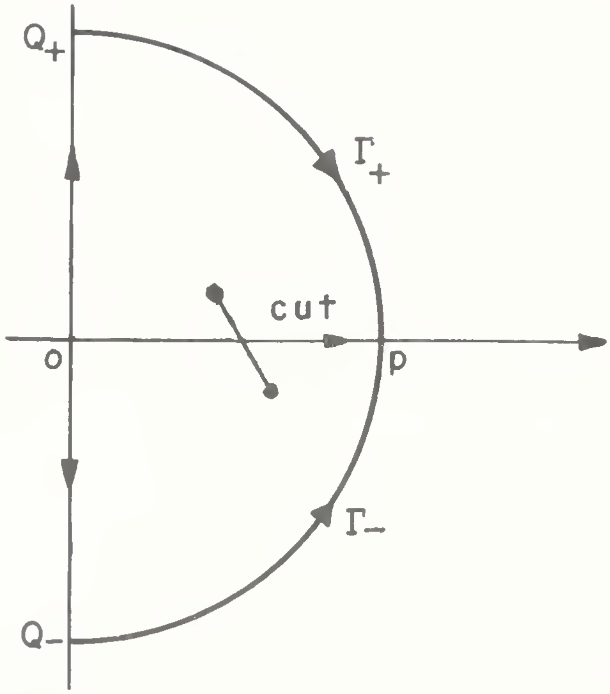

Figure 7

f_1 and f_2. The solutions of (16) are asymptotic to linear combinations of Hankel functions of one-third order. Noting that f_1 and f_2 must satisfy (17) and (22) respectively, we find that

$$(43) \qquad f_1 \sim c \left(\frac{g}{g_x}\right)^{1/2} H^{(1)}_{1/3}\left(v(x)\right)$$

$$(44) \qquad f_2 \sim c' \left(\frac{g}{g_x}\right)^{1/2} \Bigg\{ H^{(2)}_{1/3}\left(v(x)\right) \left((n^2-a^2)^{1/2}\dot{H}^{(1)}_{1/3}(v) - iZH^{(1)}_{1/3}(v)\right)_{x=x_b}$$

$$- H^{(1)}_{1/3}\left(v(x)\right) \left((n^2-a^2)^{1/2}\dot{H}^{(2)}_{1/3}(v) - iZH^{(2)}_{1/3}(v)\right)_{x=x_b}\Bigg\} .$$

Here the dot over the Hankel function means differentiation with respect to its argument. The functions $v(x)$ and $g(x)$ are defined by

$$(45) \qquad v(x) = \frac{2}{3} kg^{3/2} = k \int_{x_a}^{x} (n^2-a^2)^{1/2} \, dx.$$

From the above definition we see that g also satisfies the equation

$$(46) \qquad gg_x^2 = n^2 - a^2.$$

In (45), x_a is a turning point for (16), i.e., a root of $n(x_a) = a$.

For large k, the Wronskian W(a) may be obtained asymptotically from (43) and (44). From it we find that (42) becomes

$$(47) \qquad \Omega\, H^{(1)}_{1/3}\left[v(x_b)\right] = \left[n^2(x_b) - a^2\right]^{1/2} \dot{H}^{(1)}_{1/3}\left[v(x_b)\right] - iZ\, H^{(1)}_{1/3}\left[v(x_b)\right] = 0.$$

In (47) we have defined the operator Ω which will prove to be convenient. From the asymptotic form of W(a), $\frac{\partial W}{\partial a}$ can be calculated. Using this and (43), (44) in (41), we find

$$(48) \qquad u \sim \frac{\pi k}{6} \sum_m \frac{(gg(x_o))^{3/4}\, H^{(1)}_{1/3}\big(v(x_o)\big)\, H^{(1)}_{1/3}(v)e^{ika_m|y|} \quad \Omega H^{(1)}_{1/3}\left[v(x_b)\right]}{\left[n^2-a_m^2\right]^{1/4} \left[n^2(x_o)-a_m^2\right]^{1/4} \quad \frac{\partial}{\partial a}\, \Omega\, H^{(1)}_{1/3}\left[v(x_b)\right]} \, .$$

In each term of (48), a is to be evaluated at a_m, a root of (47).

We now simplify (48) by utilizing the Wronskian of the function $H^{(1,2)}_{1/3}\left[v(x_b)\right]$ in conjunction with (47). We then introduce the asymptotic form of $H^{(1)}_{1/3}(v)$ and $H^{(1)}_{1/3}[v(x_o)]$ and carry out the differentiation with respect to a. The expression (48) for the field then becomes

$$(49) \qquad u \sim \frac{4e^{2\pi i/3}}{\pi} \sum_m \frac{(n^2(x_b)-a_m^2)^{1/2} \exp\left(ik\left\{\int_{x_{a_m}}^{x} + \int_{x_{a_m}}^{x_o}\right\}(n^2-a_m^2)^{1/2}\,dx + ika_m|y|\right)}{(n^2-a_m^2)^{1/4}(n^2(x_o)-a_m^2)^{1/4}\left(\frac{\partial}{\partial a}\, v^2(x_b)\right) H^{(1)}_{1/3}[v(x_b)]\, \Omega\, \dot{H}^{(1)}_{1/3}[v(x_b)]} \, .$$

Let us denote the roots of (47), i.e., the values of $v(x_b)$ satisfying (47), by $\frac{2e^{-\pi i}}{3^{3/2}}\, q_m^{3/2}$. Now the roots a_m are determined by equating $v(x_b)$ to $\frac{2}{3^{3/2}}\, e^{-\pi i} q_m^{3/2}$ which yields

$$(50) \qquad k\int_{x_a}^{x_b} (n^2-a^2)^{1/2}\,dx = \frac{2}{3^{3/2}}\, e^{-\pi i}\, q_m^{3/2} \, .$$

We now solve (50) for large k in the same way as we solved (19). The result is

$$(51) \qquad a_m \sim n(x_b) + \left[6n(x_b)\right]^{-1/3} \left[k^{-1}n'(x_b)\right]^{2/3} e^{i\pi/3}\, q_m + \ldots$$

From (51), we can obtain the asymptotic forms of $\left[n^2(x_b) - a_m^2\right]^{1/2}$. If we use this and the relation between the Hankel and Airy functions in (47) this becomes the following equation for q_m

$$(52) \qquad \frac{A'(q_m)}{A(q_m)} = e^{5\pi i/6}\, z \left[\frac{k}{6n(x_b)n'(x_b)}\right]^{1/3} \, .$$

The Airy function in (52) is defined by

$$(53) \qquad A(s) = \int_0^\infty \cos(t^3 - st)\,dt.$$

We next simplify the exponentials in (49). By applying (51), we find that

$$(54) \qquad \left\{ \int_{x_{a_m}}^{x_0} + \int_{x_{a_m}}^{x} \right\} (n^2 - a_m^2)^{1/2} dx$$

$$\sim \left\{ \int_{x_b}^{x_0} + \int_{x_b}^{x} \right\} \left[n^2 - n^2(x_b) - 2.6^{-1/3}\, e^{i\pi/3} \left[k^{-1} n(x_b) n'(x_b) \right]^{2/3} q_m \right]^{1/2} dx + 2\int_{x_{a_m}}^{x_b} (n^2 - a_m^2)^{1/2} dx.$$

We now substitute for the third integral its value from (50) and expand the integrand of the first two asymptotically obtaining four integrals. If the point (x,y) is in the shadow, we can simplify the second pair of integrals by means of the equations of the diffracted ray on which (x,y) lies and the limiting ray respectively. This yields the result

$$(55) \qquad \left\{ \int_{x_{a_m}}^{x_0} + \int_{x_{a_m}}^{x} \right\} (n^2 - a_m^2)^{1/2} dx \sim \left\{ \int_{x_b}^{x_0} + \int_{x_b}^{x} \right\} \left[n^2 - n^2(x_b) \right]^{1/2} dx$$

$$- \left[k^{-1} n'(x_b) \right]^{2/3} \left[6n(x_b) \right]^{-1/3} e^{\pi i/3} q_m \left| y - y(x_b) - Y(x_b) \right|$$

$$- 4(3^{-1} q_m)^{3/2} k^{-1}.$$

Here $Y(x_b)$ and $y(x_b)$ are respectively the ordinates of the points where the limiting ray is tangent to the boundary and where the diffracted ray splits off.

We now use the relation between the Hankel and Airy functions, the differential equation for the Airy function and (52) to rewrite the term $H^{(1)}_{1/3}[v(x_b)] \;\Omega\; \dot{H}^{(1)}_{1/3}[v(x_b)]$

in (49). Inserting this result, the asymptotic forms for $\left[n^2(x_b) - a_m^2\right]^{1/2}$ and $\frac{\partial}{\partial a} v(x_b)$ and the result (55) into (49), we obtain for any point (x,y) in the shadow

$$\text{(56)} \quad u(x,y) \sim \frac{\pi e^{\pi i/3}}{4 \cdot 6^{1/3}} \left(\frac{K(x_b)}{kn(x_b)}\right)^{2/3} \frac{\exp\left(ik\left\{\int_{x_b}^{x} + \int_{x_b}^{x_o}\right\}\left[n^2 - n^2(x_b)\right]^{1/2} dx + ikn(x_b)|y|\right)}{\left(\frac{n^2}{n^2(x_b)} - 1\right)^{1/4} \left(\frac{n^2(x_o)}{n^2(x_b)} - 1\right)^{1/4}}$$

$$\cdot \sum \frac{\exp\left(e^{5\pi i/6} q_m \left[6^{-1} kn(x_b)\right]^{1/3} K^{2/3}(x_b)\, |y(x_b) - Y(x_b)| - 4i(3^{-1} q_m)^{3/2}\right)}{q_m A^2(q_m) + 3A'^2(q_m)} .$$

Here $K(x_b)$ is the curvature of the incident ray at the point of tangency to the boundary. Since the second exponential has a negative real part, the field decays exponentially with the distance $|y(x_b) - Y(x_b)|$ from the shadow boundary. The result (56) has been compared with that of the geometric theory in Part I, Section 5.

For the problems corresponding to the limiting cases $z \to \infty$ and $z = 0$, i.e., to the boundary condition $u = 0$ and $u_x = 0$, the eigenvalue equation (52) reduces to $A(q_m) = 0$ and $A'(q_m) = 0$ respectively. In these cases we can deduce the following simple asymptotic formulas for the q_m for large m.

$$\frac{2}{3^{3/2}} q_m^{3/2} = \pi\left(m + \frac{3}{4} - \frac{\delta}{2}\right) \qquad \delta = \begin{cases} 0 & u = 0 \\ 1 & u_x = 0 \end{cases}$$

In these two cases we may obtain the field u from (56) by making use of the equations for q_m. The reflection coefficient R given by (35) also simplifies, in these cases, to ∓ 1.

4. The field of a point source in a plane stratified medium with a plane boundary

We now consider the problem of a point source situated at $(x_o,0,0)$ in a plane stratified medium with a plane boundary at $x = x_b$. The index of refraction is again $n(x)$. The field u is to satisfy (1) for $x > x_b$, with the inhomogeneous term $-\frac{\delta(x-x_o)\,\delta(r)}{2\pi r}$, (22) and the three dimensional radiation condition. The field will depend only upon x and the cylindrical coordinate r.

We define $f(x,a)$, the Hankel transform of $u(x,r)$, by

$$(57) \qquad \frac{1}{2\pi} f(x,a) = \int_o^\infty u\, J_o(kar) r dr.$$

Upon taking the transform of (1) we see that f again satisfies (16) with an inhomogeneous term $-\delta(x-x_o)$ as well as (17) and (22). Therefore f is given by (26), as before. If we take the inverse transform, we obtain

$$(58) \qquad u = \frac{k^2}{2\pi} \int_o^\infty f\, J_o(kar)\, a da.$$

This can be shown to be the solution of the boundary value problem, provided that f satisfies the above conditions.

We now use the same procedures as in the case of the line source to find the asymptotic form of the field in the lit and shadow regions. First the Bessel function is replaced by its value in terms of Hankel functions. This yields the two integrals

$$(59) \qquad I^{\pm} = \frac{k^2}{4\pi} \int_o^\infty f\, H_o^{(1),(2)}(kar)\, a da.$$

To obtain the expansion for u in the lit region, we split the range of integration in (59) into the three intervals 0 to $n(x_b)$, $n(x_b)$ to n_1, and n_1 to ∞. In each of these we substitute the asymptotic expansions of $H_o^{(1),(2)}(kar)$ and the appropriate expansions for f using (29) - (33). For the interval from 0 to $n(x_b)$, we obtain for

I_1^+ an expression of the form (34) in which y is replaced by r, the multiplicative factor $\frac{i}{4\pi}$ by $\frac{1}{\pi}\left(\frac{ki}{32\pi r}\right)^{1/2}$ and the integrand is multiplied by $a^{1/2}$.

This integral can be evaluated asymptotically by the method of stationary phase, the equations for the stationary points being (36) and (37) with y replaced by r. For certain (x,r) there are two stationary values for I_1^+ and none for I_2^+, I_3^+ and I^-. Thus I^+ is given by the sum of the contributions from I_1^+. If we denote these contributions by u_i and u_r, we have $u \sim u_i + u_r$ where

$$(60) \qquad u_i \sim \frac{1}{4\pi}\left(\frac{a}{r\displaystyle\int_{x_<}^{x_>}\frac{n^2dx}{(n^2-a^2)^{3/2}}}\right)^{1/2} \frac{\exp\left(ik\displaystyle\int_{x_<}^{x_>}(n^2-a^2)^{1/2}dx + ikar\right)}{(n^2-a^2)^{1/4}\left[n(x_o)^2 - a^2\right]^{1/4}}$$

$$(61) \qquad u_r \sim \frac{R}{4\pi}\left(\frac{a}{r\left\{\displaystyle\int_{x_b}^{x_o} + \int_{x_b}^{x}\right\}\frac{n^2dx}{(n^2-a^2)^{3/2}}}\right)^{1/2} \frac{\exp\left(ik\left\{\displaystyle\int_{x_b}^{x_o} + \int_{x_b}^{x}\right\}(n^2-a^2)^{1/2}dx+ikar\right)}{(n^2-a^2)^{1/4}\left[n(x_o)^2 - a^2\right]^{1/4}} .$$

The remaining integrals $I_2^\pm$ and $I_3^\pm$ can be evaluated in the manner outlined after equation (39). For all points in the lit region the result is the same as that just described - $u \sim u_i + u_r$. This result has been compared with that of the geometrical theory in Part I, Section 6.

Let us now consider points x,y in the shadow region. As before, we will rewrite (59) in a way which will turn out to be convenient in this region. From Figure 7, we see that an equation of the form (40) holds with e^{ikay} replaced by a $H_o^{(1)}(kar)$. The remarks following (40) concerning the case $y \geq 0$ also apply here.

In the same way as before, we find that u is given asymptotically by a residue series of the form (41) with $e^{ika_m|y|}$ replaced by a $H_o^{(1)}(kar)$ and the factor ik replaced by $\frac{1}{2}\,ik^2$. The a_m are again the roots of (42) which are given asymptotically by (51). We utilize this result, (43), (44), and the asymptotic expansion of the

Hankel functions to simplify the residue series. Then, using (52)-(55) we further simplify the result. We finally find that for any point (x,r) in the refraction shadow the field u is given asymptotically by

$$(62)\quad u(x,r) \sim 6^{-1/3}\left(\frac{\pi e^{\pi i/6}}{32r}\right)^{1/2}\left(\frac{K^{2/3}(x_b)}{\left[kn(x_b)\right]^{1/6}}\right)\frac{\exp\left(ik\left\{\int_{x_b}^{x}+\int_{x_b}^{x_o}\right\}\left[n^2-n^2(x_b)\right]^{1/2}dx+ikn(x_b)r\right)}{\left(\frac{n^2}{n^2(x_b)}-1\right)^{1/4}\left(\frac{n^2(x_o)}{n^2(x_b)}-1\right)^{1/4}}$$

$$\cdot\sum\frac{\exp\left(e^{5\pi i/6}q_m\left[6^{-1}kn(x_b)\right]^{1/3}K^{2/3}(x_b)\left[r(x_b)-R(x_b)\right]-4i(3^{-1}q_m)^{3/2}\right)}{q_mA^2(q_m)+3A'^2(q_m)}.$$

Here $R(x_b)$ and $r(x_b)$ are respectively the r coordinates of the point of tangency of the limiting ray (i.e. the grazing incident ray) and the point of shedding of the diffracted ray through (x,r). This result has been compared with that of the geometric theory in Part I, Section 6.

5. The field of a line source in a cylindrically stratified medium with a circular boundary

Let us consider a cylindrically stratified medium for which $n = n(r)$ and which is bounded internally by a circle $r = b$. Let there be a line source parallel to the z-axis and situated at $(r_o,0)$. Again, the field will be independent of z. We require that the index $n(r)$ be monotonic and approach n_1 as r tends to ∞. For $r > b$, the field u satisfies (1) with an inhomogeneous term $-r^{-1}\delta(r-r_o)\delta(\theta)$ on the right side. On the boundary u must satisfy (22) with $\frac{\partial u}{\partial x}$ replaced by $\frac{\partial u}{\partial r}$ and at infinity the radiation condition (23).

Since u has to be even and periodic in θ, we assume that the field can be written in the form

$$u = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} f(r,m)\, e^{im\theta}. \tag{63}$$

The function f(r,m) is expressible as the Fourier integral

$$f(r,m) = \int_0^{2\pi} u e^{-im\theta}\, d\theta. \tag{64}$$

We now apply this finite Fourier transform to (1) with the right side properly modified. We find that the function $r^{1/2} f(r,m)$ satisfies (16) with a^2 replaced by $(m^2 - \frac{1}{4})(kr)^{-2}$ and with the inhomogeneous term $-r^{-1/2}\, \delta(r-r_0)$ on the right side.

The solution to this equation satisfying (17) and modified (22) is given by an expression of the form (26). However, in it $x_>$, $x_<$, and a must be replaced respectively by $r_>$, $r_<$ and m and the entire expression must be multiplied by r_0^{-1}. With this value for f(r,m), we can verify that (63) is the solution to our boundary problem. However, to obtain an asymptotic expansion for u, it is convenient to rewrite (63) by using a Watson transformation.

By Cauchy's residue theorem, we can transform the sum in (63) into the contour integral

$$u = -\frac{1}{4\pi i} \int_{C_1 + C_2} \frac{f(r,\alpha) e^{i\alpha(\theta-\pi)}}{\sin \pi\alpha}\, d\alpha. \tag{65}$$

Here, C_1 and C_2 are the two straight lines Im $\alpha = \pm$ constant in the neighborhood of the real axis. The direction of integration along C_1 is from left to right and for C_2 from right to left. Using the fact that f is even in α we can write (65) as

$$u = -\frac{1}{2\pi i} \int_{C_1} \frac{f(r,\alpha) \cos \alpha(\theta-\pi)}{\sin \pi\alpha}\, d\alpha. \tag{66}$$

To put (66) in a form which will yield the field in the lit region, we set $\cos \alpha(\theta-\pi) = -ie^{i\alpha\theta}\sin \pi\alpha + e^{i\pi\alpha}\cos \alpha\theta$. We then move the contour for the first summand, which now has no poles on the real axis, down to the real axis and then let $\alpha = ka$ in both integrals. As a consequence, we have

$$(67) \qquad u = \frac{k}{2\pi}\int_{-\infty}^{\infty} f(r,ka)e^{iak\theta}da - \frac{k}{2\pi i}\int_{C_1} \frac{f(r,ka)e^{i\pi ka}\cos ak\theta}{\sin \pi ak}\, da \;.$$

The first integral, being of the form (27), can be treated as in the previous two sections. If we divide the positive a-axis into the three intervals 0 to n(b), n(b) to n_1 and n_1 to ∞, we find that

$$(68) \qquad I_1^+ \sim \frac{i}{4\pi}\int_0^{n(b)} \frac{\exp\left(ik\int_{r_<}^{r_>}\left(n^2 - \frac{a^2}{r^2}\right)^{1/2} dr + ika\theta\right) + R\exp\left(ik\left\{\int_b^r + \int_b^{r_o}\right\}\left(n^2 - \frac{a^2}{r^2}\right)^{1/2} dr + ika\theta\right)}{\left[r^2n^2 - a^2\right]^{1/4}\left[r_o^2n^2(r_o) - a^2\right]^{1/4}}\, da \,.$$

Here

$$(69) \qquad R = \frac{\left[b^2n^2(b)-a^2\right]^{1/2} + bZ}{\left[b^2n^2(b)-a^2\right]^{1/2} - bZ} \,.$$

We now apply the method of stationary phase to I_1^+ and find two stationary points for certain values of (r,θ). The integrals $I_{1,2}^+$ and I^-, for these values of (r,θ), do not have stationary points. The contributions to the field by the first integral of (67) is thus the sum of the contributions from the two stationary points of I_1^+. If we designate these contributions by u_i and u_r, we get the geometric field $u_g \sim u_i + u_r$ where

$$(70) \qquad u_i \sim \frac{\exp\left(ik\int_{r_<}^{r_>}(n^2 - \frac{a^2}{r^2})^{1/2} dr + ika\theta + \frac{i\pi}{4}\right)}{\left[r_o^2n^2(r_o)-a^2\right]^{1/4}\left[r^2n^2-a^2\right]^{1/4}\left(8\pi k\int_{r_<}^{r_>}\frac{rn^2\, dr}{(r^2n^2-a^2)^{3/2}}\right)^{1/2}} \,,$$

$$(71)\quad u_r \sim \frac{R\, \exp\left(ik \left\{ \int_b^r + \int_b^{r_o} \right\} \left(n^2 - \frac{a^2}{r^2} \right)^{1/2} dr + ika\theta + \frac{i\pi}{4} \right)}{\left[r_o^2 n^2(r_o) - a^2 \right]^{1/4} \left[r^2 n^2 - a^2 \right]^{1/4} \left(8\pi k \left\{ \int_b^r + \int_b^{r_o} \right\} \frac{rn^2 dr}{(r^2 n^2 - a^2)^{3/2}} \right)^{1/2}} .$$

The remaining integrals $I^+_{2,3}$, I^- are handled as in Section 3 and similar results are obtained.

We now show that the second integral of (67) yields the diffracted field u_d. We first make the change of variables $\alpha = ka$. Then we take the resulting integral over a finite segment of C_1. This is equivalent to an integral along a semicircle in the upper half plane plus the residues of the integrand. Again, it can be shown that the integral over the semicircle tends to zero as its radius approaches infinity. We are left with the residue series

$$(72)\quad u_d = \frac{k}{r_o} \sum_m \frac{e^{i\pi k a_m} \cos k a_m \theta \;\; f_1(r_>, ak) f_2(r_<, ak)}{\frac{\partial}{\partial a} W(a_m)} .$$

Here, the a_m are the zeros of the Wronskian $W(a)$.

The analysis now proceeds along previous lines. The eigenfunctions f_1, f_2 are replaced by their asymptotic representations (43), (44). However, the function v is now defined by

$$(73)\qquad v(r) = \int_{r_a}^{r} \left(n^2 - \frac{a^2}{r^2} \right)^{1/2} dr.$$

The quantity r_a is a zero of $r^2 n^2 - a^2$ and corresponds to a turning point. Instead of (46), g now satisfies the equation $g g_r^2 = n^2 - a^2$. Then, asymptotically for large k the zeros a_m are given by (47) with $\left[n^2(x_b) - a^2 \right]^{1/2}$ replaced by $\left[n^2(b) - a^2 b^{-2} \right]^{1/2}$ and with v defined by (73). These eigenvalues can be approximated as in Section 3. Instead of (51), we obtain

$$(74)\qquad a_m \sim bn(b) + q_m e^{\pi i/3} \left[\frac{b}{6n(b)} \right]^{1/3} \left[\frac{bn'(b) + n(b)}{k} \right]^{2/3} .$$

The q_m satisfy (52) with the factor $n^{-1/3}(x_b)\left[n'(x_b)\right]^{-1/3}$ replaced by $\left[b\left\{bn'(b)+n(b)\right\}^{-1}\right]^{1/3}$.

We now can carry out approximations and simplifications similar to those following equations (47) and (51) in Section 3. As a consequence we obtain for the diffracted field at any point (r,θ), the result

$$(75)\qquad u_d(r,\theta) \sim \frac{\pi e^{i\pi/3}}{4\cdot 6^{1/3}}\left[\frac{b\,K(b)+1}{kbn(b)}\right]^{2/3} \frac{\exp\left[ik\left\{\int_b^r + \int_b^{r_o}\right\}\left[n^2 - \frac{b^2n^2(b)}{r^2}\right]^{1/2} dr\right]}{\left[\frac{r^2n^2}{b^2n^2(b)} - 1\right]^{1/4}\left[\frac{r_o^2n^2(x_o)}{b^2n^2(b)} - 1\right]^{1/4}}$$

$$\cdot \sum_m \frac{\exp\left[-4i(3^{-1}q_m)^{3/2}\right]}{\left[1-\exp\left\{2\pi ikbn(b)+2\pi e^{5\pi i/6}S_m\right\}\right]\left[q_mA^2(q_m)+3A'^2(q_m)\right]}$$

$$\cdot \left\{ \exp\left[ikbn(b)(2\pi+\theta) + e^{5\pi i/6}q_m\left\{6^{-1}kbn(b)\right\}^{1/3}\left\{b\,K(b)+1\right\}^{2/3}(2\pi + \theta_b - \Psi_b)\right]\right.$$

$$\left. + \exp\left[ikbn(b)(2\pi - \theta) + e^{5\pi i/6}q_m\left\{6^{-1}kbn(b)\right\}^{1/3}\left\{b.K(b)+1\right\}^{2/3}(2\pi - 2\theta + \theta_b - \Psi_b)\right]\right\}.$$

In this expression, $K(b)$ is the curvature of the limiting ray at the point of tangency (b,Ψ_b) to the circular boundary. (b,θ_b) is the point on the boundary where the diffracted ray splits off. The quantity S_m is defined by

$$(76)\qquad S_m = q_m\left[6^{-1}kbn(b)\right]^{1/3}\left[b\,K(b) + 1\right]^{2/3}.$$

Another formula for the diffracted field can be derived from (66) in exactly the same way. We find instead of (75)

$$(77) \quad u_d(r,\theta) \sim \frac{\pi e^{\pi i/3}}{4 \cdot 6^{1/3}} \left[\frac{b\,K(b)+1}{kbn(b)} \right]^{2/3} \frac{\exp\left[ik \left\{ \int_b^r + \int_b^{r_o} \right\} \left[n^2 - \frac{b^2 n^2(b)}{r^2} \right]^{1/2} dr \right]}{\left[\frac{r^2 n^2}{b^2 n^2(b)} - 1 \right]^{1/4} \left[\frac{r_o^2 n^2(r_o)}{b^2 n^2(b)} - 1 \right]^{1/4}}$$

$$\cdot \sum_m \frac{\exp\left[-4i(3^{-1} q_m)^{3/2} \right]}{\left[1 - \exp\left\{ 2\pi ikbn(b) + 2\pi e^{5\pi i/6} S_m \right\} \right] \left[q_m A^2(a_m) + 3A'^2(q_m) \right]}$$

$$\cdot \left\{ \exp\left[ikbn(b)\theta + e^{5\pi i/6} q_m \left\{ 6^{-1} kbn(b) \right\}^{1/3} \left\{ b\,K(b)+1 \right\}^{2/3} (\theta_b - \Psi_b) \right] \right.$$

$$\left. + \exp\left[ikbn(b)(2\pi - \theta) + e^{5\pi i/6} q_m \left\{ 6^{-1} kbn(b) \right\}^{1/3} \left\{ b\,K(b)+1 \right\}^{2/3} (2\pi - 2\theta + \theta_b - \Psi_b) \right] \right\} .$$

This form is valid in the refraction shadow whereas (75) holds in the lit region. The two sums in each expression are the modes of the cylinder. From (77), we can get the result (56) of Section 3. We set $r = x+b$, $r_o = x_o+b$, $\theta = b^{-1}y$ and let b tend to ∞. On the other hand, by setting $n' = 0$, we obtain the asymptotic form of the diffracted field due to a line source in a homogeneous medium.

The results (70), (71) and (77) have been compared with the geometrically constructed fields of Part I.

APPENDIX

The mode expansion of the field due to a line source in a plane stratified medium with a plane boundary is obtained by applying a Watson transformation. We now show that the branch-cut integral and integral along the pure imaginary axis which arise in connection with the transformation are asymptotically zero. Also, the integral along the semi-circle tends to zero with increasing radius.

By using (43) and (44) we can express the integrand of (28) asymptotically for large k. We then obtain for the integral the expression

$$\text{(A1)} \qquad \frac{ki}{12} \int\limits_{\Gamma_{\pm} + OQ_{\pm} + B_{\pm}} \frac{[gg(x_o)]^{3/4} e^{ikay}}{[n^2 - a^2]^{1/4} [n^2(x_o) - a^2]^{1/4}} \Bigg[H^{(2)}_{1/3}(v(x_>)) - H^{(1)}_{1/3}(v(x_>)) \frac{\Omega\, H^{(2)}_{1/3}(v(x_b))}{\Omega\, H^{(1)}_{1/3}(v(x_b))} \Bigg] H^{(1)}_{1/3} v(x_<) da \,.$$

A similar result involving e^{-ikay} holds. However, the branch cut on the real axis occurs for only one of these expressions, depending on how we close the contour in the complex a-plane.

Due to the multi-valuedness of $v(x_o)$ (or $v(x)$ and $v(x_b)$) its argument will differ on the opposite sides of the cut B*. Consider first $v(x_o)$. We subdivide the real positive axis into the intervals 0 to $n(0)$, $n(0)$ to $n(x_o)$, $n(x_o)$ to n_1 and n_1 to ∞. Then values correspond to particular turning points of (16), Part II. Now the value $n(x_o)$ is a branch point of $v(x_o)$. Because of the monotonicity of $n(x)$ (see Figure 1a), it follows that to every value of a in the interval $n(0)$ to n, corresponds a real turning point x_a. In particular for a in the interval $n(0)$ to $n(x_o)$, the turning point $x_a < x_o$. Therefore, the quantity $(n^2 - a^2)^{1/2}$ is real and the argument of $v(x_o)$ is zero. For those a in the interval $n(x_o)$ to n_1, the turning point $x_a > x_o$ so that $(n^2 - a^2)^{1/2}$ is pure imaginary. Near the turning point $v \sim \frac{2}{3} n'(x_a)(x - x_a)^{3/2}$. Thus, the argument of $v(x_o)$ for these values of a is $-3\pi/2$.

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*The cut actually extends from $-n(x_>)$ to $n(x_>)$ on the real axis. For the analysis of (A1), we consider only the part on the positive real axis.

Corresponding to the intervals $(0, n(0))$ and (n_1, ∞) there are no real turning points so that the argument of $v(x_o)$ in the first interval is zero and in the second $-3\pi/2$. On the other side of the cut, its argument from 0 to $n(x_o)$ must be -3π.

In the same way, we can show that the arguments of $v(x)$ and $v(x_b)$ are zero and -3π on either side of the branch cuts determined by their respective branch points of $n(x)$ and $n(x_b)$ and $-3\pi/2$ for values of a beyond the branch point.

With the previous remarks, we find that the expression

$$[gg(x_o)]^{3/4} \, [n^2 - a^2]^{-1/4} \, [n^2(x_o) - a^2]^{-1/4} \tag{A2}$$

has on one side of the branch cut up to the branch point $n(x_>)$ the argument zero and on the other side the argument -2π. For values of $a > n(x_>)$, its argument becomes $-\pi/2$ and $-3\pi/2$ respectively.

By making use of the relations

$$H^{(1),(2)}_{1/3}(x) = -H^{(1),(2)}_{1/3}(xe^{\pm 3\pi i}) \tag{A3}$$

$$\dot{H}^{(1),(2)}_{1/3}(x) = \dot{H}^{(1),(2)}_{1/3}(xe^{\pm 3\pi i}) \tag{A4}$$

we now can show that (A1) is zero. In the interval $(n(x_b), n(x_a))$, for example, the two products resulting from the multiplication of (A2) with the

Hankel functions of arguments $v(x)$ and $v(x_o)$ are positive on one side of the cut. Since on the other side, (A2) has an argument of -2π and then Hankel functions have arguments of -3π, application of (A3) and (A4) shows that these products are still positive. On the other hand the argument of $\Omega H^{(2)}_{1/3} / \Omega H^{(1)}_{1/3}$ doesn't change in this interval. Since the paths of integration are traversed in opposite directions, the integral over this interval is zero. In the same way, we can show that (A1) vanishes over the remaining portions of B.

On the imaginary axis the argument of v changes from zero to -3π in going from Im $a > 0$ to Im $a < 0$ and that of (A2) changes from zero to -2π. Therefore, if we set $a = -iz$ and utilize (A3) and (A4), we find

$$\text{(A5)}\quad \int_{OQ_-} f\, e^{-ikay} dy \sim \frac{k}{12} \int_{OQ_-} \frac{[gg(x_o)]^{3/4}\, e^{-kzy}}{[n^2 + z^2]^{1/4}\, [n^2(x_a) + z^2]^{1/4}}$$

$$H^{(1)}_{1/3}\left[v(x_<)e^{-3\pi i}\right] \left[H^{(2)}_{1/3}\left[v(x_>)e^{-3\pi i}\right] - H^{(1)}_{1/3}\left[v(x_>)e^{-3\pi i}\right] \frac{\Omega\, H^{(2)}_{1/3}\left[v(x_b)e^{-3\pi i}\right]}{\Omega\, H^{(1)}_{1/3}\left[v(x_b)e^{-3\pi i}\right]} \right] dz$$

$$= \frac{k}{12} \int \frac{[gg(x_o)]^{3/4}\, e^{-kzy}}{[n^2 + z^2]^{1/4} [n^2(x_o) + z^2]^{1/4}}\, H^{(1)}_{1/3}[v(x_<)] \left[H^{(2)}_{1/3}[v(x_>)] \right.$$

$$\left. - H^{(1)}_{1/3}[v(x_>)]\, \frac{\Omega\, H^{(2)}_{1/3}[v(x_b)]}{\Omega\, H^{(1)}_{1/3}[v(x_b)]} \right] dz$$

$$\sim -\int_{OQ_+} f\, e^{ikay}\, dy.$$

Thus, the sum of the two integrals along the imaginary axis is asymptotically equal to zero.

If we now expand the Hankel functions in (A1) asymptotically for large $|a|$, we find that the integrand behaves like

$$(A6) \qquad e^{-k(x_>-x_<)\rho\cos\theta} + Re^{-k(2x_b-x_o-x)\rho\cos\theta} .$$

In (A6), $a = \rho e^{i\theta}$ and $R = O(1)$ for large ρ. Along Γ_+, $0 \le \arg a \le \frac{\pi}{2}$ so that

$$(A7) \qquad \int_{\Gamma_+} = O\left\{\int_0^{\pi/2}\left(e^{-k(x_>-x_<)\rho\cos\theta}\,d\theta + Re^{-k(2x_b-x_o-x)\rho\cos\theta}\right)d\theta\right\} .$$

But

$$(A8) \qquad \int_0^{\pi/2} e^{-k(x_>-x_<)\rho\cos\theta}\,d\theta \le \frac{c}{\rho}\left[1 - e^{-k\rho(x_>-x_<)}\right] .$$

For $x \neq x_o$, this integral tends to zero as $\rho \to \infty$. In the same way, the second integral of (A7) can be shown to approach zero. When $x = x_o$, the term corresponding to the first exponential of (A7) reduces to unity. However, instead of using the bound of unity on $|e^{ikay}|$, we estimate its integral and this yields an expression given by an equation such as (A8). As a result, the integral along Γ_+ tends to zero for all x as $\rho \to \infty$.

Finally, we note that the analogous manipulations for the problems of Sections 4 and 5 of Part II may be justified in the same way.

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